Elementary Example of L’Hôpital’s Rule

We begin by applying L’Hôpital’s rule to a problem we could have solved earlier:

\[
\lim_{x \to 1} \frac{x^{10} - 1}{x^2 - 1}.
\]

We listed some categories of limits at the beginning of the course; this falls into the category of “interesting limits” because if we just plug in \( x = 1 \) we get \( \frac{0}{0} \). This is called an indeterminate form.

To find the limit using techniques we already know, we’d do the following:

\[
\lim_{x \to 1} \frac{x^{10} - 1}{x - 1} = \lim_{x \to 1} \frac{(x^{10} - 1)/(x - 1)}{(x^2 - 1)/(x - 1)}.
\]

We could calculate \( (x^{10} - 1)/(x - 1) \) using long division, but that’s a long calculation. We can find this limit more quickly using calculus.

We’ve used calculus to understand a fraction in indeterminate form when we studied the difference quotient. If \( f(x) = x^{10} - 1 \), then \( f(1) = 0 \) and the difference quotient is:

\[
\frac{f(x) - f(1)}{(x - 1)} = \frac{x^{10} - 1}{x - 1}.
\]

We know from our studies of difference quotients that:

\[
\lim_{x \to 1} \frac{f(x) - f(1)}{(x - 1)} = f'(1).
\]

We conclude that:

\[
\lim_{x \to 1} \frac{x^{10} - 1}{x - 1} = f'(1) = 10.
\]

Our expression:

\[
\frac{x^{10} - 1}{x^2 - 1} = \frac{(x^{10} - 1)/(x - 1)}{(x^2 - 1)/(x - 1)}
\]

describes a ratio of difference quotients, so if \( g(x) = x^2 - 1 \) this line of reasoning tells us that:

\[
\lim_{x \to 1} \frac{x^{10} - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x^{10} - 1)/(x - 1)}{(x^2 - 1)/(x - 1)}
\]

\[
= \lim_{x \to 1} (x^{10} - 1)/(x^2 - 1)
\]

\[
= \frac{f'(1)}{g'(1)}
\]

\[
= \frac{10}{2} = 5.
\]

Dividing by \( x - 1 \) and interpreting the fraction as a ratio of difference quotients enabled us to solve the problem by taking two easy derivatives and saved us from a lengthy exercise in long division.