## Rate of Growth of $e^{px}$

When we looked at $\lim_{x\to 0^+} x\ln x$ we found that the value of the limit was 0, so $x$ shrinks to 0 faster than $\ln x$ grows to negative infinity. The next two examples illustrate similar rate properties, which will be important when we study improper integrals and elsewhere.

**Example:** $\displaystyle\lim_{x\to\infty} xe^{-px}, \quad (p>0)$

The expression $xe^{-px}$ is a product, not a ratio, so we need to rewrite it before we use l'Hôpital's rule. We choose to rewrite it as $\frac{x}{e^{px}}$. This is of the form $\frac{\infty}{\infty}$, so we can use l'Hôpital's rule to calculate:

$$\begin{aligned}
\lim_{x\to\infty} xe^{-px} &= \lim_{x\to\infty} \frac{x}{e^{px}} \\
&= \lim_{x\to\infty} \frac{1}{pe^{px}} \qquad \text{(l'Hop)} \\
&= \frac{1}{\infty} \\
&= 0.
\end{aligned}$$

We conclude that when $p>0$, $x$ grows more slowly than $e^{px}$ as $x$ goes to infinity.

**Example:** $\displaystyle\lim_{x\to\infty} \frac{e^{px}}{x^{100}} \quad (p>0)$

This example doesn't give us much more information, but it's good practice. The value of this limit gives us information about the relative rates of growth of $e^{px}$ and $x^{100}$.

The expression $\displaystyle\lim_{x\to\infty} \frac{e^{px}}{x^{100}}$ is of the form $\frac{\infty}{\infty}$, so we can use l'Hôpital's rule again. In fact, there are two ways we could use l'Hôpital's rule. The slow way looks like:

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^{px}}{x^{100}} &= \lim_{x\to\infty} \frac{pe^{px}}{100x^{99}} \qquad \text{(l'Hop)} \\
&= \lim_{x\to\infty} \frac{p^2e^{px}}{100\cdot 99x^{98}} \qquad \text{(l'Hop)} \\
&= \lim_{x\to\infty} \frac{p^3e^{px}}{100\cdot 99\cdot 98x^{97}} \qquad \text{(l'Hop)} \\
&\vdots
\end{aligned}$$

We could apply l'Hôpital's rule 100 times and we'd eventually get an answer.

The clever way is to rewrite the expression as follows:

$$\lim_{x\to\infty} \frac{e^{px}}{x^{100}} = \left(\lim_{x\to\infty} \frac{e^{px/100}}{x}\right)^{100}$$

$$
\begin{aligned}
&= \left(\lim_{x\to\infty} \frac{\frac{p}{100}e^{px/100}}{1}\right)^{100} \qquad \text{(l'Hop)} \\
&= \left(\lim_{x\to\infty} \frac{p\cdot e^{px/100}}{100}\right)^{100} \\
&= \infty
\end{aligned}
$$

In this example $\lim_{x\to a} \frac{f'(a)}{g'(a)} = \infty$, another possible outcome of l'Hôpital's rule. We conclude that $e^{px}$ grows faster than $x^{100}$ when $p$ is positive. In fact, $e^{px}$ grows faster than *any* polynomial in $x$; exponential functions grow faster than powers of $x$.