l’Hôpital’s Rule, Continued

In keeping with the spirit of “dealing with infinity” we look at an application of l’Hôpital’s rule to a limit of the form \( \frac{\infty}{\infty} \). In other words, as \( x \) approaches \( a \) we have:

- \( f(x) \to \infty \)
- \( g(x) \to \infty \)
- \( \frac{f'(x)}{g'(x)} \to L \)

and so we can conclude that:

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = L.
\]

(Recall that \( a \) and \( L \) may be infinite.)

Rates of Growth

We apply this to “rates of growth”; the study of how rapidly functions increase. We know that the functions \( \ln x \) and \( x^2 \) both go to infinity as \( x \) goes to infinity, and that \( x^2 \) increases much more rapidly than \( \ln x \). We can formalize this idea as follows:

If \( f(x) > 0 \) and \( g(x) > 0 \) as \( x \) approaches infinity, then

\( f(x) \ll g(x) \) as \( x \to \infty \) means \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \).

(Read \( f(x) \ll g(x) \) as “\( f(x) \) is a lot less than \( g(x) \”).) In our example, \( f(x) = \ln x \) and \( g(x) = x^2 \). If we use l’Hôpital’s rule to evaluate \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\ln x}{x^2} \) we get:

\[
\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0.
\]

We conclude that \( \ln x \ll x^2 \) as \( x \to \infty \).

If \( p > 0 \) then:

\( \ln x \ll x^p \ll e^x \ll e^{x^2} \) as \( x \to \infty \).

Rates of Decay

“Rates of decay” are rates at which functions tend to 0 as \( x \) goes to infinity. Again our new notation comes in handy; if \( p > 0 \) then:

\[
\lim_{x \to \infty} \frac{1}{\ln x} \gg \frac{1}{x^p} \gg \frac{1}{e^x} \gg e^{-x} \gg e^{-x^2} \text{ as } x \to \infty.
\]