\[ \lim_{x \to \infty} (x^{1/x}) \]

Use an extension of l'Hôpital's rule to compute \( \lim_{x \to \infty} (x^{1/x}) \).

**Solution**

This calculation is very similar to the calculation of \( \lim_{x \to 0^+} x^x \) presented in lecture, except that instead of the indeterminate form \( 0^0 \) we instead have \( \infty^0 \).

As before, we use the exponential and natural log functions to rephrase the problem:

\[ x^{1/x} = e^{\ln x^{1/x}} = e^{\frac{\ln x}{x}}. \]

Thus, \( \lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\frac{\ln x}{x}} \). Since the function \( e^t \) is continuous,

\[ \lim_{x \to \infty} e^{\frac{\ln x}{x}} = e^{\lim_{x \to \infty} \frac{\ln x}{x}}. \]

We can now focus our attention on the limit in the exponent; \( \lim_{x \to \infty} \frac{\ln x}{x} \) is in the indeterminate form \( \frac{\infty}{\infty} \), so l'Hôpital's rule is applicable.

\[
\frac{\ln x}{x} = \frac{1/x}{1} \quad \text{(provided the limit exists)}
\]

\[
= \frac{0}{1}
\]

\[
= 0
\]

We conclude that \( \lim_{x \to \infty} x^{1/x} = e^{\lim_{x \to \infty} \frac{\ln x}{x}} = 1. \)

This implies that the \( n^{\text{th}} \) root of \( n \) approaches 1 as \( n \) approaches infinity.