Example: $\int_1^\infty \frac{dx}{x^p}$

We know that $\int_1^\infty \frac{dx}{x}$ diverges. Next we’ll find $\int_1^\infty \frac{dx}{x^p}$ for any value of $p$; we’ll see that $p = 1$ is a borderline when we do this calculation.

\[
\int_1^\infty \frac{dx}{x^p} = \int_1^\infty x^{-p}dx \\
= \left. \frac{x^{-p+1}}{-p+1} \right|_1^\infty \\
= \frac{\infty^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} \\
= \frac{-p+1}{-p+1 + \frac{1}{p-1}}
\]

Remember that the $\infty$ in this expression is shorthand for “a number approaching infinity”.

When we think about raising a very large number to the $p + 1$ power we see that there are two cases that split exactly at $p = 1$. When $p = 1$, the exponent is zero and so is the denominator; the expression doesn’t make any sense. For all other values of $p$ the expression makes sense and the value of the integral depends on whether $-p + 1$ is positive or negative.

\[
\frac{\infty^{-p+1}}{-p+1} \text{ is infinite when } -p + 1 > 0
\]

and

\[
\frac{\infty^{-p+1}}{-p+1} \text{ is zero when } -p + 1 < 0.
\]

Check this yourself — this is the sort of problem that will be on the exam.

**Conclusion:** Combining this with our previous example we see that:

$\int_1^\infty \frac{dx}{x^p}$ diverges if $p \leq 1$

and

$\int_1^\infty \frac{dx}{x^p}$ converges to $\frac{1}{p-1}$ if $p > 1$.

Notice that when $p = 1$ our formula for the antiderivative is wrong; the antiderivative is $\ln x$ and not $\frac{x^{-p+1}}{-p+1}$. We really needed to do three separate calculations to compute the value of this integral: one for $p < 1$, one for $p = 1$ and one for $p > 1$. 

