Overview of Improper Integrals

Now let’s contrast the two types of improper integrals we’ve looked at — one in which $x$ goes to infinity and one in which $x$ approaches a point of singularity.

We have just considered functions like:

$$\frac{1}{x^{1/2}} << \frac{1}{x} << \frac{1}{x^2} \quad \text{as } x \to 0^+.$$  

Conversely,

$$\frac{1}{x^{1/2}} >> \frac{1}{x} >> \frac{1}{x^2} \quad \text{as } x \to \infty.$$  

In general, we found that improper integrals of functions smaller than $\frac{1}{x}$ converge while improper integrals of functions larger than or equal to $\frac{1}{x}$ diverge. Whether a function is smaller or larger than $\frac{1}{x}$ depends on the function and on what limit we’re taking:

$$\frac{1}{x^{1/2}} << \frac{1}{x} << \frac{1}{x^2} \quad \text{as } x \to 0^+. \quad \text{integral converges}$$

$$\frac{1}{x^{1/2}} >> \frac{1}{x} >> \frac{1}{x^2} \quad \text{as } x \to \infty. \quad \text{integral diverges}$$

As shown in Figure 1, the graph of $f(x) = \frac{1}{x}$ is symmetric to itself by a reflection across the line $y = x$. The total area under the curve to the right of $x = 1$ is infinite and so is the area under the curve between $x = 0$ and $x = 1$.

The graph of $y = \frac{1}{x^{1/2}}$ lies below that of $y = \frac{1}{x}$ on the left and above $\frac{1}{x}$ on the right. (See Figure 3.) The total area under the curve of $y = \frac{1}{x^{1/2}}$ to the right of $x = 1$ is infinite, but the area under the curve between $x = 0$ and $x = 1$ is 2. (See Figure 2.)
\[
\int_0^1 \frac{dx}{x^{1/2}} = 2
\]

\[
\int_1^\infty \frac{dx}{x^{1/2}} = \infty
\]

Figure 2: Area under the graph of \( y = \frac{1}{x^{1/2}} \).

\[
\int_0^1 \frac{dx}{x^{1/2}} = 2
\]

\[
\int_1^\infty \frac{dx}{x^{1/2}} = \infty
\]

Figure 3: Graph of \( y = \frac{1}{x} \) superimposed on graph of \( y = \frac{1}{x^{1/2}} \).

Compare this to the area under the graph of \( y = \frac{1}{x^{3/2}} \). Here the area to the right of 1 is finite (2) and the area between 0 and 1 is infinite. (See Figure 4.)

By comparing the sizes of the vertical and horizontal “tails” of the functions we can get a geometric sense of the difference between convergent and divergent indefinite integrals.
\[
\int_{1}^{\infty} \frac{dx}{x^2} = 2
\]

Figure 4: Area under the graph of \( y = \frac{1}{x^2} \).

\[
\int_{0}^{1} \frac{dx}{x^2} = \infty
\]

\[
\int_{1}^{\infty} \frac{dx}{x^2} = 2
\]

Figure 5: Graph of \( y = \frac{1}{x} \) superimposed on graph of \( y = \frac{1}{x^2} \).