

An Improper Integral of the Second Kind

Suppose we want to calculate:

$$\int_0^{\infty} \frac{dx}{(x-3)^2}.$$

In calculating $\int_0^{\infty} \frac{dx}{(x-3)^2}$ you must worry about two pieces — say $\int_0^5 \frac{dx}{(x-3)^2}$ and $\int_5^{\infty} \frac{dx}{(x-3)^2}$.

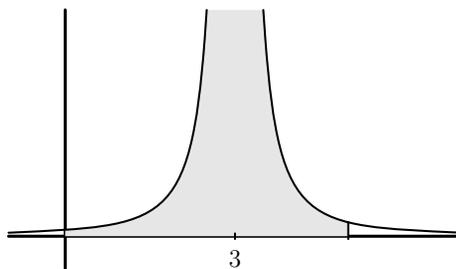


Figure 1: $\int_0^5 \frac{dx}{(x-3)^2} = \infty$.

The singularity in the graph of $y = \frac{dx}{(x-3)^2}$ is comparable to that of $y = \frac{1}{x^2}$ near $x = 0$. The area under the graph of $y = \frac{dx}{(x-3)^2}$ between 0 and 5 is infinite.

However, $\int_5^{\infty} \frac{dx}{(x-3)^2}$ converges.

Unfortunately, the total $\int_0^{\infty} \frac{dx}{(x-3)^2}$ diverges because it's the sum of a divergent indefinite integral and a convergent one.

If you failed to notice the singularity at $x = 3$ you might have calculated the value of this integral to be finite, the same way we calculated the false value of $\int_{-1}^1 \frac{dx}{x^2}$. In that case you might still be saved from a terrible error by noticing that you calculated a negative value for the integral of a function that is everywhere positive.

Question: How do we know when and where to look for “bad spots” in an integrand?

Answer: If the either limit of integration is infinite, check the limits as x goes to infinity and/or minus infinity. Also, check any singularity, like x going to 3 in the integrand $\frac{1}{(x-3)^2}$. In other words, look in all the places where the integrand may be infinite. Once you've identified the problem areas, you can focus on each one separately by splitting the domain of integration into parts.

Specific things to watch for are negative exponents — anything of the form $\frac{dx}{x^n}$ goes to infinity as x approaches zero. This includes all the integrals we computed using partial fractions; whenever there's something in the denominator that can be zero, there's a singularity.

MIT OpenCourseWare
<http://ocw.mit.edu>

18.01SC Single Variable Calculus
Fall 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.