Operations on Power Series Related to Taylor Series

In this problem, we perform elementary operations on Taylor series – term by term differentiation and integration – to obtain new examples of power series for which we know their sum. Suppose that a function \( f \) has a power series representation of the form:

\[
f(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - c)^n
\]

convergent on the interval \((c - R, c + R)\) for some \( R \). The results we use in this example are:

- (Differentiation) Given \( f \) as above, \( f'(x) \) has a power series expansion obtained by by differentiating each term in the expansion of \( f(x) \):

\[
f'(x) = a_1 + a_2(x - c) + 2a_3(x - c) + \cdots = \sum_{n=1}^{\infty} na_n(x - c)^{n-1}
\]

- (Integration) Given \( f \) as above, \( \int f(x) \, dx \) has a power series expansion obtained by by integrating each term in the expansion of \( f(x) \):

\[
\int f(x) \, dx = C + a_0(x - c) + \frac{a_1}{2}(x - c)^2 + \frac{a_3}{3}(x - c)^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - c)^{n+1}
\]

for some constant \( C \) depending on the choice of antiderivative of \( f \).

Questions:

1. Find a power series representation for the function \( f(x) = \arctan(5x) \). (Note: \( \arctan x \) is the inverse function to \( \tan x \).)

2. Use power series to approximate

\[
\int_0^1 \sin(x^2) \, dx
\]

(Note: \( \sin(x^2) \) is a function whose antiderivative is not an elementary function.)

Solution:

For question (1), we know that \( \arctan x \) has a simple derivative: \( \frac{1}{1 + x^2} \), which then has a power series representation similar to that of \( \frac{1}{1 - x} \), where we substitute \(-x^2\) for \( x \). Hence:

\[
\frac{d}{dx} \arctan(5x) = \frac{5}{1 + 25x^2} = 5 \sum_{n=0}^{\infty} (-25x^2)^n = \sum_{n=0}^{\infty} (-1)^n 5^{2n+1} x^{2n},
\]

where the second equality above follows from the familiar geometric series representation for \( \frac{1}{1 - x} \). The last equality presents a cleaner final form after straightforward algebraic simplification. Thus
to obtain a power series expression for \( \arctan x \) we may integrate this power series expression term by term. This gives:

\[
\arctan(5x) = C + 5x - \frac{5^3}{3}x^3 + \cdots = C + \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{2n+1},
\]

and we may solve for \( C \) by comparing both sides of the equality for any value of \( x \). Choosing \( x = 0 \), we see that \( \arctan(x) = 0 \) and all non-constant terms of the power series are 0, hence \( C = 0 \) as well.

For question (2), we have seen that \( \sin(x) \) has a power series expansion:

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.
\]

Using a change of variable (replacing \( x \) by \( x^2 \) in the power series above), we have the power series expansion

\[
\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}.
\]

Now taking the indefinite integral of both sides, we obtain a power series representation for the antiderivative of \( \sin(x^2) \):

\[
\int \sin(x^2) \, dx = \frac{1}{3} x^3 - \frac{1}{7} \frac{x^7}{3!} + \frac{1}{11} \frac{x^{10}}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3} \frac{x^{4n+3}}{(2n+1)!}.
\]

The power series expression is valid for any real number \( x \) since the power series for \( \sin(x) \), and hence \( \sin(x^2) \) converged for all \( x \).

To approximate the definite integral, we may use as many terms of the series as we like. For example, using only the first non-zero term would give:

\[
\int_0^1 \sin(x^2) \, dx \approx \frac{1}{3} x^3 \bigg|_{x=0}^{x=1} = \frac{1}{3}.
\]

The first two non-zero terms gives:

\[
\int_1^2 \sin(x^2) \, dx \approx \left( \frac{1}{3} x^3 - \frac{1}{7} \frac{x^7}{3!} \right) \bigg|_{x=0}^{x=1} = \left( \frac{1}{3} - \frac{1}{42} \right) = \frac{13}{42}.
\]

Using a numerical integration on a computer-algebra system, we find that the answer is approximately .31026... while \( 13/42 = .309524 \). We can improve this estimate by using more terms in the power series.