

Welcome back to recitation. In this video, I'd like us to do some basic manipulation of some power series that you saw in the lecture.

So I have four problems here, and what I'd like us to do, is figure out what function each of these series represents. Now, you can assume that the x-values that we are going to insert into this function are only x-values that let the sums converge. So you don't have to worry about anything to do with convergence of these sums. Just assume the sums converge, that we've picked good x-values. OK? And what I'd like you to have in the end, for (a), (b), (c), and (d), is something like, this sum is equal to a specific function. It should not include a sum anymore.

So why don't you work on those for a bit, pause the video, and when you're done, restart the video, and I'll come back, and show you how I work with them.

All right. Welcome back. Well, hopefully these were a little bit fun for you. I always liked them, the first time I saw them, to see how one could manipulate these series.

So again, what we're trying to do is figure out what function these series represent. We're assuming convergence. And we're going to try and manipulate them to look like things we already know. So I'm going to start. I'm just going to go straight through (a), (b), (c), and (d). And I'm going to rewrite the problem each time, because I don't want to keep coming back.

So (a), we had the sum  $n$  equals 0 to infinity,  $x$  to the  $n$  plus 2 over  $n$  factorial.

Well, this looks very close to something we know. It looks a lot like the one, the function  $e$  to the  $x$ . The difference is that  $e$  to the  $x$  just has a power  $x$  to the  $n$ . But the good news is that I am really close to that. What I really have in the numerator is  $x$  to the  $n$  times  $x$  squared. Because this  $x$  to the  $n$  plus 2, I can write as  $x$  to the  $n$  times  $x$  squared. So every term has an  $x$  squared.

So as was mentioned in the lecture, you can treat this really like polynomials. In some way, you can factor this out. So I can rewrite this as-- well, I'll rewrite it in two steps. 0 to infinity of  $x$  squared,  $x$  to the  $n$  over  $n$  factorial. So  $x$  squared times  $x$  to the  $n$  gives me  $x$  to the  $n$  plus 2. Right?

But I can actually now, because this belongs to every sum, I can pull that all the way out in front. Right? And if I pull that all the way out in front, if I move this out to the front, I have an  $x$  squared times this sum. Well, what is the sum? The sum is  $e$  to the  $x$ . It's  $n$  equals 0 to infinity of  $x$  to the  $n$  over  $n$  factorial. That's just  $e$  to the  $x$ . So this function is  $x$  squared  $e$  to the  $x$ . That's what (a) is.

If you were worried about it, you could write  $e^x$  as a series, and then you could multiply through by  $x^2$ , and see if that's what you get. But you'll see, that is indeed how this problem works.

OK. Let's look at (b). OK, (b) is equal to the sum-- that's a weird summation sign, sorry about that-- (b) is equal to the sum  $n=2$  to infinity of  $x^n$ . That's what we wanted to look at. Well, this looks very close to the geometric series, but it's missing some terms. Right? The geometric series starts at  $n=0$  to infinity. But the point I want to make here is that I can rewrite this as the geometric series, and then I can take away what I've added in. OK?

So these do not agree right now. This equals sign is not true yet. But what do I notice? I notice that this one has two more terms at the beginning than this one has. What are those terms? Those are when  $n=0$  and when  $n=1$ . Right? There's no  $n=0$  or  $n=1$  here. The formulas are exactly the same. Right?  $x^n$ ,  $x^n$ . They both go to infinity, but one is starting at  $n=2$ , and one's starting at  $n=0$ .

Again, I want to remind you. Why did I bother to write this thing at all? Because this is a function we know. But this equals sign is not true right now. Right? I have to get rid of the extra stuff I added. What did I add? I added-- let me get this as a sum. What did I add on? I added on  $x^0$  and  $x^1$ . So now I guess I should have written subtract here, right? I should subtract  $x^0$ , and I should subtract  $x^1$ .

I've already said it once, but I'm going to, just to make sure everybody follows, say it one more time. I now-- I've taken the summation I started with, which went from  $n=2$  to infinity. I added two more terms to the sum. I made it go from 0 to infinity. So to keep equality, I subtracted off the two things I added in. I subtracted off  $x^0$  and  $x^1$ .

So now, what was the point of this part of the exercise? Well, the point is that I know what this sum is. That's the geometric series. That's  $\frac{1}{1-x}$ . And this  $x^0$ , it's  $-1$ ;  $x^1$ ,  $-x$ . So this I broke up into something I knew as a power series, and then other pieces that I had added in to make it look like something I knew. I had to subtract those off.

So that's the idea. That one, if you wanted to go on and simplify further, you could do that. But I'm willing to leave it just as is. 'Cause the idea was really this part up here, and then translating it down to this.

All right. Let me write (c) again. The sum  $n=0$  to infinity,  $\frac{x^n}{n!} + x^n$ . OK. So the sum  $n=0$  to infinity,  $\frac{x^n}{n!} + x^n$ . And the point to recognize here is, again, as Professor Jerison mentioned, you're really treating these almost like polynomials. So you're taking a series, and you're adding these terms up. But really, what you can think about this, is the two separate series added up. And so this is the sum from  $n=0$  to infinity of  $\frac{x^n}{n!}$ , which we know, plus the sum from  $n$

equals  $0$  to infinity of  $x$  to the  $n$ , which we know. The first one is just  $e$  to the  $x$ , and the second one, we've been dealing a lot with these already today, is just  $1$  over  $1$  minus  $x$ .

So the point I want to make, is if you have this convergent series, you can split it up into pieces over the sum. And these two are both convergent, we know, separately, and so we can write what they are.  $x$  to the  $n$  over  $n$  factorial from  $0$  to infinity is  $e$  to the  $x$ ,  $x$  to the  $n$  from  $n$  equals  $0$  to infinity is  $1$  over  $1$  minus  $x$ .

So now we just have one more. Let me write that one.  $(d)$  was summation  $n$  equals minus  $1$  to infinity  $x$  to the  $n$  plus  $1$ . All right. This was not meant to scare you, but it was meant to test your understanding of sigma notation. So the problem is, we're starting at  $n$  equals minus  $1$ , but  $x$  is starting-- the exponent on  $x$  is at  $n$  plus  $1$ . So I want to write it in some form that I know. Well, let's try and get the subscript to be  $0$ . so what I'm going to do, is I'm going to change the name of the subscript. I'm going to let it be  $m$  equals  $0$  to infinity. OK? And I want  $m$  to count up by  $1$  just the way  $n$  counts up.

So notice what I did. When  $m$  equals  $0$ ,  $n$  equals negative  $1$ . That's what we've set up as the first term. That means  $n$  plus  $1$  is equal to  $0$  when  $m$  equals  $0$ . And since I'm going up by one every time in my summation, my iteration, here that means that for every  $m$  I have, I just have to take  $n$  plus  $1$  to figure out what  $m$  is. So I should have said it the other way. For every  $n$  I have, I just have to add  $1$  to get  $m$ . OK?

So here, if I start at negative  $1$ , I start here at  $0$ . The next term here is  $0$ ,  $n$  equals  $0$ , but the next term here is  $m$  equals  $1$ . The next term here would be  $n$  equals  $1$ , and the next term here would be  $m$  equals  $2$ . So they're just off by  $1$ , but they're still catching every index.

Now, I don't have to change what's up here. Because infinity-- if I add  $1$  to it, I'm still going off to infinity. So I don't have to change what's up here. But I do want to write the formula, or the formula I have inside the sum in terms of  $m$ . But I've already got it. Because I know  $m$  is always equal to  $n$  plus  $1$ , so I can replace this  $n$  plus  $1$  by an  $m$ .

And now we know what that one is. Right?  $m$  equals  $0$  to infinity of  $x$  to the  $m$ . That's our geometric series. So it actually, even though I wrote it in kind of a funny way, it was actually just still the geometric series. I just moved the indices a little bit to make sure we could play with those.

So the idea here, the whole point of this exercise, just to recognize how you can manipulate these series a little bit, so that if they're in a form that looks kind of like one of the functions you know, you can see if it actually is, you know, a product of something with the function you know, or the sum of two functions you know, or maybe one of the functions you know is a power series, and then you have to drop off a couple of terms. So they each sort of demonstrate a different idea of how you can manipulate these convergent power series, based on functions you already know.

So I'll stop there.