

**PROFESSOR:** Welcome back to recitation. In this video I want to finish up our work with the ratio test, and in particular in this video, I'd like to address something that Professor Jerison talked about. When he was talking about these Taylor series, he was talking about the radius of convergence at some point. And he didn't go into it very specifically, but he was essentially saying, for some values of  $x$ -- you'll have a series that has  $x$  to the  $n$  in it-- for some values of  $x$  it will converge, and for some values it will diverge.

So let me just remind you of something you already know. So, if we consider the series  $x$  to the  $n$ . Right? This is the geometric series and we know that this is equal to  $1$  over  $1$  minus  $x$  if the absolute value of  $x$  is less than  $1$ . Right? And so we know that this series is finite. This sum converges. To whatever value. If I plug in  $x$  here, it converges when absolute value of  $x$  is less than  $1$ . And it diverges when absolute value of  $x$  is bigger than  $1$ . And we're not going to address when the absolute value of  $x$  equals  $1$ . We won't address that case.

But, the point I want to make it is that for some values this series converges, and for some values this series diverges. And those values can give us the radius of convergence. So the radius of convergence of this series is actually  $1$ , because  $x$  goes from  $0$  up to  $1$ , and then from  $0$  down to  $1$ . If you think about it, radius might be a confusing term, but can think about it as a circle in one dimension less than maybe you usually think about it as a circle. If this is the number line, we're going from  $0$  up to  $1$  and down to minus  $1$ . So you're going in one direction up to  $1$  and one direction down to minus  $1$ . So this is radius  $1$ .  $r$  is equal to  $1$ .

So what I want to do is figure out how I can use the ratio test to tell me the radius of convergence of other series, other power series, besides this one. OK? So we're really interested in for what values of  $x$  do these power series converge. OK? And how we're going to do that, is we're going to directly use the ratio test. And one thing that I mentioned in the other case, in the other ratio test video, is I said you want to have your terms be positive. And so just to make this easy on ourselves, when we do the ratio test we're just going to take the absolute value of the ratio. And that will be sufficient for our purposes, to determine the radius of convergence. I don't want to go into anything more complicated than that. So we'll see, as we do these examples, we're just gonna, we're just gonna take the absolute value of the ratio.

So let's actually start off right away with doing some examples. And each time I do an example, I'm asking the following question. Actually, I was going to say for what values of  $x$ ,

but that's not quite true. I just want to know what is the radius of convergence for each power series? So I'm not going to write it up every time, but this is the question we want to be thinking about. So I'm going to write down some power series and we're going to see what is the radius of convergence for each of those.

I was going to write, find the values of  $x$  for which the power series converge, but that's not quite true because sometimes it will actually converge on one or both of the endpoints. We're not going to deal with that one. But sometimes it will converge on one or the other of the endpoints. So I'm going to be asking the question, find the radius of convergence.

So let's do an example, and I'll show you how it applies to the ratio test. Then we'll go from there and do some other examples. So let's consider the series  $x$  over  $2$  to the  $n$ . This will be an easy one. Now the ratio test told us that we examine a certain limit. We examine the limit as  $n$  goes to infinity. It said  $a_{n+1}$  over  $a_n$ . Well, if we think of this whole thing as a  $a_n$ , then  $a_{n+1}$  is  $x$  over  $2$  to the  $n+1$ . And then we have to divide by  $x$  over  $2$  to the  $n$ .

So let me just make sure we understand that. The way we, the way we want to think about this is as if the  $a_n$  is actually a function of  $x$  that also depends on  $n$ . OK, so  $a_n$ , we think of it as  $x$  over  $2$  raised to the  $n$ . So  $a_{n+1}$  is  $x$  over  $2$  raised to the  $n+1$ .  $a_n$  is  $x$  over  $2$  raised to the  $n$ . OK, and so this looks exactly like the ratio test we had before, I told you I'm going to put absolute values on it. And so let me simplify this. I put the division sign maybe kind of funny 'cause I wanted to have it in a row there. I apologize if that made it more confusing. So this equals, well I have  $x$  to the  $n+1$  over  $x$  to the  $n$  times  $2$  to the  $n$ -- this comes to the numerator because it's in the denominator of the denominator-- over-- and this goes to the denominator--  $2$  to the  $n+1$ .

Now what is this? This whole thing converges to what?  $2$  to the  $n$  over  $2$  to the  $n+1$  simplifies to  $1/2$ , and  $x$  to the  $n+1$  over  $x$  the  $n$  simplifies to  $x$ . So this is equal to absolute value of  $x$  over  $2$ . Now we haven't shown how the ratio test is going to help us find the radius of convergence.

And here's where we're going to see it working. If we want to know that, we want to draw a conclusion about the convergence of this series. We know that if this limit is less than  $1$ , that the series definitely converges. So if I take my limit when I'm all done, and I set it less than  $1$ , and I solve for absolute value of  $x$ , then that will tell me exactly what the radius of convergence

is. Let me go through that one more time.

We knew that if we had this series in terms of a sub  $n$ , and we looked at the ratio and we took the limit of these ratios, and in the limit the value was strictly less than 1, we know the series converges. So now we have this thing. It's in terms of  $x$ , but ultimately it's the same process. And we get to a place where we know the limit. It depends on  $x$ . And so the limit will be less than 1 exactly where this thing is less than 1. Right? So it's now in terms of  $x$ . This less than 1, I'm putting in at the end. This is where the ratio test is happening.

So what does this tell us? Where does this converge? It converges when absolute value of  $x$  over 2 is less than 1, which means absolute value of  $x$  is less than 2. Right? I just multiply by 2 'cause absolute value of 2 is 2. I don't have to worry about if 2 is negative.

So the absolute value of  $x$  is less than 2. That tells me the radius of convergence is actually 2. Now this shouldn't surprise us really. Because this is, in fact, a geometric series, and we know that the usual geometric series converges when the absolute value of  $x$  is less than 1. So it's not surprising, and actually fits in with what we already know, that if this thing on the inside, if its absolute value is less than 1, that the thing will converge. So that that means absolute value of  $x$  has to be less than 2. So again, hopefully this fits in with the knowledge we already have. Hopefully it makes sense to you, what we're trying to do. And so let's do another couple of examples to make this, make this solidify.

All right, what were my other examples? OK, let's do this one.  $x$  to the  $n$  over  $n$  factorial. All right, so now our a sub  $n$ , when we look at this geometric-- geometric? This is not geometric-- when we look at the series, our a sub  $n$  is going to be  $x$  to the  $n$  over  $n$  factorial. And so when we take our whole limit, the thing that's going to happen is we're going to have something in terms of  $x$  at the end. Our goal is to find for what values of  $x$  is that thing we have less than 1? Again this is where we're using the ratio test.

So let's look at this thing. We want to look at the limit as  $n$  goes to infinity. Well, a sub  $n$  plus 1 is going to be  $x$  to the  $n$  plus 1 over the quantity  $n$  plus 1 factorial. I'm going to put the absolute value over there. Now I'm going to multiply it by 1 over a sub  $n$ . So this is actually a sub  $n$ , so 1 over that is  $n$  factorial over  $x$  to the  $n$ .

Now we have to be a little careful here. I want to make sure everybody understands something.  $n$  factorial is  $n$  times  $n$  minus 1 times  $n$  minus 2 times  $n$  minus 3 all the way down.  $n$  plus 1 factorial is  $n$  plus 1 times  $n$  times  $n$  minus 1 times  $n$  minus 2 all the way down. So  $n$

plus 1 factorial-- I'll just write this here--  $n + 1$  factorial equals  $n + 1$  times  $n$  factorial. That's very important that you recognize that, OK? Because the division of factorials is a little more complicated than the division of straight polynomials or something like this. All right?

Because when I take this limit, what am I going to get? I'm going to get  $x$  to the  $n + 1$  over  $x$  to the  $n$ . That's going to give me one  $x$ . And then  $n$  factorial divided by  $n + 1$  factorial, well based on the fact that this is equal to  $n + 1$  times  $n$  factorial, the  $n$  factorials divide, and I'm left with  $x$  over  $n + 1$ . That's actually what this limit equals. Let me go through that one more time.

This part is easy,  $x$  to the  $n + 1$  over  $x$  to the  $n$  gives me the  $x$ . And then  $n$  factorial over  $n + 1$  factorial is just  $1$  over  $n + 1$ . And I have to be careful because I wrote equals, but there's still an  $n$  here. So let me erase that. I'm not gonna, I don't think-- my studio audience didn't say anything yet, but I don't think they were going to let me get away with that.

This is the limit as  $n$  goes to infinity of  $x$  over  $n + 1$ . Now what is that? Well as  $n$  goes to infinity, for any fixed  $x$  I pick-- we have to be careful here-- for any fixed  $x$  I pick, as  $n$  goes to infinity, this quantity is equal to  $0$ . OK? If I were moving  $x$  around, if I were moving  $x$  with  $n$ , this will be a problem, but  $x$  is fixed. When I do this sum, I fix my  $x$  at the beginning. So for any fixed  $x$ ,  $n + 1$  is getting arbitrarily large. So  $x$  over  $n + 1$  is getting arbitrarily small. So this limit is  $0$ . This is strictly less than  $1$ .

What does this mean? For any  $x$  I pick, this whole thing is less than  $1$ . Right? Any fixed  $x$ , this ratio is always less than  $1$ . What does that mean? That means the radius of convergence is infinite. So the radius is infinity. OK. The radius of convergence is actually infinite. What series is this? This is actually the Taylor series for  $e$  to the  $x$ . So we know that the Taylor series for  $e$  to the  $x$ , that this converges for any  $x$ . That's a nice thing. So now we've used the ratio test to tell us how the Taylor series behaves for a function that we know. OK, we know its radius of convergence is infinite. So that's pretty nice. All right, let's do maybe one or two more examples, depending on how much room I have.

OK, let's try this one.  $x$  to the  $n$  over  $n$  times  $2$  to the  $n$ . All right, so this is another power series we have. And we want to know, if I want to plug in some value of  $x$  and I want to take the sum, will that converge for that particular value of  $x$ ? We want to know what values of  $x$  can I plug in. All right so again, we're going to look at the radius of-- find the radius of convergence based on the ratio test. So the  $n$  plus first term is  $x$  to the  $n + 1$  over  $n + 1$   $2$  to the  $n + 1$ . And

then I have to multiply by the  $a_{n+1}$ 's, the  $a_n$  term, or multiply by  $1$  over that, sorry. I'm dividing by  $a_n$ , so I get  $n$  times  $2$  to the  $n$  over  $x$  to the  $n$ .

All right, this gives me an  $x \cdot n + 1$  over  $n$ , let me just write out actually what we get here,  $2$  to the  $n$  over  $2$  to the  $n + 1$  gives me an over  $2$ . And then  $n$  over  $n + 1$ . This is positive, so I don't have to worry about anything there. The limit as  $n$  goes to infinity absolute value of  $x$  over  $2$  times  $n$  over  $n + 1$ , what's that equal to? Well this, as  $n$  goes to infinity, is equal to  $1$ , so it's absolute value of  $x$  over  $2$ .

Let me again remind you what we are doing. We're saying I want to know the radius of convergence for this series. So I want to know what radius, so what values of  $x$  can I put in to make this series converge? I might miss the endpoints, but beyond that, what values of  $x$  make this series converge? So I'm looking at the ratio test. I took the ratio test, I got it all the way to a place where I have something in terms of  $x$ . As long as that thing is less than  $1$ , I'm golden. As long as that thing is less than  $1$ , the series converges.

So again, I actually get another thing where the absolute value of  $x$  is less than  $2$ . All right? I probably should have picked a different number there. What do you think would happen if this was a  $7$ ? Well everything would've been the same except this would have been a  $7$ , and the radius of convergence would have been  $7$ . So I should have picked a different number there, so we had a different radius of convergence, but you can see how that works. And then, I'm gonna do just--

I have room, I'm going to do one more example. 'Cause this one's a good one to do, also. OK, I'm actually going to do another series that we know,  $x$  to the  $2n$  over  $2n$  factorial. I'm doing this one for a particular reason, to help us deal with when some things, when the exponents and the factorials get a little more complicated.

I want to also point out what series is this. You should know what series this is. And I know, without thinking, that it's either sine or cosine. I guess I should say it's starting at  $n$  equals  $0$  to infinity. And then I might get nervous and say well, which one is it? Well what's the first term?  $x$  to the  $0$  is  $1$  over  $0$  factorial is  $1$ . So it looks like the first term is  $1$ , so that makes me know it's cosine. Right? If the first term were  $x$ , I'd know it was sine. So if I get nervous. So this, where it converges, is equal to  $\cos x$ .

Let's look at this one. All right, we got, now this you have to be a little more careful because it's  $2$  times the quantity  $n + 1$  over  $2n + 2$  factorial. So that's  $n + 1$  times  $2$  is  $2n + 2$

times  $2n$  factorial over  $x$  to the  $2n$ . All right this is where it might get a little tricky. This is  $2n$  plus 2 divided by  $2n$ . So this is going to be the limit as  $n$  goes to infinity of  $x$  squared, the absolute value of  $x$  squared which is just  $x$  squared again, times this thing. So let's figure out what this thing is.

OK? What is  $2n$  factorial? I'm going to write out the first couple terms. OK, that's going to be  $2n$  times  $2n - 1$  times, that's all the way down to 1. And then what's this? This is  $2n$  plus 2 times  $2n$  plus 1 times  $2n$  times, all the way down to 1. So we see, this is what I was talking about earlier actually, also. Was that I've added 2 to this. So it's not surprising I get 2 more terms, and then I'm down to  $2n$  factorial again. Right? So this  $2n$  factorial. This is  $2n$  factorial. So the  $2n$  factorial here divides with the  $2n$  factorial here, and I'm left with these here. I get  $1$  over  $2n$  plus 2 times  $2n$  plus 1.

Now what happens as  $n$  goes to infinity? Obviously this ratio goes to 0. So I'm actually in another case similar to the one I saw with  $e$  to the  $x$ . Is that this goes to 0. So the limit is actually equal to 0, which is always less than 1. Any value of  $x$  I pick is less than 1. And so this series actually converges for any  $x$  that I pick. So this is another case where I have the radius of convergence is actually infinite.

OK, so we'll probably give you some problems where you have some other things happening. 'Cause you can actually get the radius convergence is 0. You can actually get that the only place it converges is at  $x$  equals 0. So you might actually get the limit as  $n$  goes to infinity is infinity. I didn't give you an example of that, but that's a case where whatever  $x$  value you put in, your limit is still bigger than 1. Then you would always get a diverges, except when  $x$  is 0. OK, so that's another thing that you can run into.

But, the point I want to make is that this is going to let us determine, you know, at least the radius over which these  $x$  values converge. And it also helps us if we know certain things about a function, to tell us where this function is actually equal to the series that we're dealing with. So I think that's where I'll stop, and I hope this was informative.