18.02 Practice Exam 1 A

Problem 1. (15 points)
A unit cube lies in the first octant, with a vertex at the origin (see figure).

a) Express the vectors $\overrightarrow{OQ}$ (a diagonal of the cube) and $\overrightarrow{OR}$ (joining $O$ to the center of a face) in terms of $\hat{i}$, $\hat{j}$, $\hat{k}$.

b) Find the cosine of the angle between $\overrightarrow{OQ}$ and $\overrightarrow{OR}$.

Problem 2. (10 points)
The motion of a point $P$ is given by the position vector $\overrightarrow{R} = 3 \cos t \hat{i} + 3 \sin t \hat{j} + t \hat{k}$. Compute the velocity and the speed of $P$.

Problem 3. (15 points: 10, 5)

a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$; then $\det(A) = 2$ and $A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & a & b \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$; find $a$ and $b$.

b) Solve the system $AX = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

c) In the matrix $A$, replace the entry 2 in the upper-right corner by $c$. Find a value of $c$ for which the resulting matrix $M$ is not invertible.

For this value of $c$ the system $MX = 0$ has other solutions than the obvious one $X = 0$: find such a solution by using vector operations. (Hint: call $U$, $V$ and $W$ the three rows of $M$, and observe that $MX = 0$ if and only if $X$ is orthogonal to the vectors $U$, $V$ and $W$.)

Problem 4. (15 points)
The top extremity of a ladder of length $L$ rests against a vertical wall, while the bottom is being pulled away. Find parametric equations for the midpoint $P$ of the ladder, using as parameter the angle $\theta$ between the ladder and the horizontal ground.

Problem 5. (25 points: 10, 5, 10)

a) Find the area of the space triangle with vertices $P_0 : (2, 1, 0)$, $P_1 : (1, 0, 1)$, $P_2 : (2, -1, 1)$.

b) Find the equation of the plane containing the three points $P_0$, $P_1$, $P_2$.

c) Find the intersection of this plane with the line parallel to the vector $\vec{V} = (1, 1, 1)$ and passing through the point $S : (-1, 0, 0)$.

Problem 6. (20 points: 5, 5, 10)

a) Let $\overrightarrow{R} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the position vector of a path. Give a simple intrinsic formula for $\frac{d}{dt}(\overrightarrow{R} \cdot \overrightarrow{R})$ in vector notation (not using coordinates).

b) Show that if $\overrightarrow{R}$ has constant length, then $\overrightarrow{R}$ and $\overrightarrow{V}$ are perpendicular.

c) let $\overrightarrow{A}$ be the acceleration: still assuming that $\overrightarrow{R}$ has constant length, and using vector differentiation, express the quantity $\overrightarrow{R} \cdot \overrightarrow{A}$ in terms of the velocity vector only.