3. Double Integrals

3A. Double integrals in rectangular coordinates

3A-1

a) Inner: \(6x^2y + y^2\) \(y=-1\) to \(y=1\); Outer: \(4x^3\) from \(0\) to \(2\).

b) Inner: \(-u \cos t + \frac{1}{2} t^2 \cos u\) \(t=0\) to \(t=\pi\); Outer: \(u^2 + \frac{1}{2} \pi^2 \sin u\) \(u=0\) to \(u=\pi/2\).

\(\frac{3}{4} \pi^2\).

c) Inner: \(x^2 y^2 \sqrt{2}\) \(x^2 = x^6 - x^3\); Outer: \(\frac{1}{2} x^7 - \frac{1}{4} x^4\) \(x=1\) to \(x=\frac{1}{4}\) to \(-\frac{3}{28}\).

d) Inner: \(v \sqrt{u^2 + 4}\) \(u\) to \(u^2 + 4\) \(v=0\) to \(v=1\); Outer: \(\frac{1}{3} (u^2 + 4)^{3/2}\) \(u=0\) to \(u=\pi\).

3A-2

a) (i) \(\int \int_R dy \, dx = \int_{-2}^{2} \int_{-x}^{x} dy \, dx\) (ii) \(\int \int_R dx \, dy = \int_{0}^{1} \int_{-y}^{y} dx \, dy\)

b) i) The ends of \(R\) are at \(0\) and \(2\), since \(2x - x^2 = 0\) has \(0\) and \(2\) as roots.

\(\int \int_R dy \, dx = \int_{0}^{1} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \, dx\)

ii) We solve \(y = 2x - x^2\) for \(x\) in terms of \(y\): write the equation as \(x^2 - 2x + y = 0\) and solve for \(x\) by the quadratic formula, getting \(x = 1 \pm \sqrt{1 - y}\).

Note also that the maximum point of the graph is \((1,1)\) (it lies midway between the two roots \(0\) and \(2\)). We get

\(\int \int_R dx \, dy = \int_{0}^{1} \int_{-\sqrt{1-y}}^{\sqrt{1-y}} dx \, dy\),

c) (i) \(\int \int_R dy \, dx = \int_{0}^{2} \int_{0}^{x} dy \, dx + \int_{0}^{2} \int_{\sqrt{4-x^2}}^{x} dy \, dx\)

(ii) \(\int \int_R dx \, dy = \int_{0}^{\sqrt{2}} \int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \, dy\)

d) Hint: First you have to find the points where the two curves intersect, by solving simultaneously \(y^2 = x\) and \(y = x - 2\) (eliminate \(x\)).

The integral \(\int \int_R dy \, dx\) requires two pieces; \(\int \int_R dx \, dy\) only one.

3A-3

a) \(\int \int_R dA = \int_{0}^{2} \int_{0}^{1-x/2} x \, dy \, dx\);

Inner: \(x(1 - \frac{1}{3}x)\) Outer: \(\frac{1}{2} x^2 - \frac{1}{8} x^3\) \(x=2\) to \(x=0\).

\(\frac{2}{3}\).
b) \[ \int_{R} (2x + y^2) \, dA = \int_{0}^{1} \int_{0}^{1-y^2} (2x + y^2) \, dx \, dy \]

Inner: \[ x^2 + y^2 \int_{0}^{1-y^2} \, dx = 1 - y^2; \quad \text{Outer: } y - \frac{1}{3}y^3 \int_{0}^{1} = \frac{2}{3}. \]

\[ \int_{R} y \, dA = \int_{0}^{1} \int_{y-1}^{1-y} y \, dx \, dy \]

Inner: \[ xy \int_{y-1}^{1-y} \, dx = y[(1 - y) - (y - 1)] = 2y - 2y^2; \quad \text{Outer: } y^2 - \frac{2}{3}y^3 \int_{0}^{1} = \frac{1}{3}. \]

3A-4 a) \[ \int_{R} \sin^2 x \, dA = \int_{-\pi/2}^{\pi/2} \int_{0}^{\cos x} \sin^2 x \, dy \, dx \]

Inner: \[ y \sin^2 x \int_{0}^{\cos x} = \cos x \sin^2 x; \quad \text{Outer: } \frac{1}{3} \sin^2 x \int_{0}^{\pi/2} - \pi / 2 = \frac{1}{3} (1 - (-1)) = \frac{2}{3}. \]

\[ \int_{R} xy \, dA = \int_{0}^{1} \int_{x^3}^{x} (xy) \, dy \, dx. \]

Inner: \[ \frac{1}{2}xy \int_{x^3}^{x} = \frac{1}{2} (x^3 - x^5); \quad \text{Outer: } \frac{1}{2}x^4 \int_{0}^{1} = \frac{1}{24}. \]

b) The function \( x^2 - y^2 \) is zero on the lines \( y = x \) and \( y = -x \), and positive on the region \( R \) shown, lying between \( x = 0 \) and \( x = 1 \).

Therefore

\[ \text{Volume} = \int_{R} (x^2 - y^2) \, dA = \int_{0}^{1} \int_{x^3}^{x} (x^2 - y^2) \, dy \, dx. \]

Inner: \[ x^2y \int_{x^3}^{x} = \frac{4}{3}x^3; \quad \text{Outer: } \frac{1}{3}x^4 \int_{0}^{1} = \frac{1}{3}. \]

3A-5 a) \[ \int_{0}^{2} \int_{x}^{2} e^{-y^2} \, dy \, dx = \int_{0}^{2} \int_{0}^{y} e^{-y^2} \, dx \, dy = \int_{0}^{2} e^{-y^2} y \, dy = -\frac{1}{2} e^{-y^2} \int_{0}^{2} = \frac{1}{2} (1 - e^{-4}) \]

b) \[ \int_{0}^{1} \int_{0}^{1} \frac{e^u}{u} \, du \, dt = \int_{0}^{1} \int_{0}^{u} \frac{e^u}{u} \, du \, dt = \int_{0}^{1} u e^u \, du = (u - 1)e^u \int_{0}^{1} = 1 - \frac{1}{2} \sqrt{e} \]

c) \[ \int_{0}^{1} \int_{x^3}^{1} \frac{1}{1 + u^4} \, du \, dx = \int_{0}^{1} \int_{0}^{u^{1/3}} \frac{1}{1 + u^4} \, dx \, du = \int_{0}^{1} \frac{u^{1/3}}{1 + u^4} \, du = \frac{1}{4} \ln(1 + u^4) \int_{0}^{1} = \ln 2 / 4. \]

3A-6 0; \quad 2 \int_{S} e^x \, dA, S = \text{right half of } R; \quad 4 \int_{Q} x^2 \, dA, Q = \text{first quadrant} \]

0; \quad 4 \int_{Q} x^2 \, dA; \quad 0

3A-7 a) \( x^4 + y^4 \geq 0 \Rightarrow \frac{1}{1 + x^4 + y^4} \leq 1 \)

b) \[ \int_{R} \frac{x \, dA}{1 + x^2 + y^2} \leq \int_{0}^{1} \int_{0}^{1} \frac{x}{1 + x^2} \, dx \, dy = \frac{1}{2} \ln(1 + x^2) \int_{0}^{1} = \ln 2 / 2 < \frac{7}{2}. \]
3B. Double Integrals in polar coordinates

3B-1

a) In polar coordinates, the line \( x = -1 \) becomes \( r \cos \theta = -1 \), or \( r = -\sec \theta \). We also need the polar angle of the intersection points; since the right triangle is a 30-60-90 triangle (it has one leg 1 and hypotenuse 2), the limits are (no integrand is given):

\[
\int_0^{4\pi/3} \int_{-\sec \theta}^{2} dr \, d\theta = \int_{2\pi/3}^{2} \int_{-\sec \theta}^{2} dr \, d\theta.
\]

c) We need the polar angle of the intersection points. To find it, we solve the two equations \( r = \frac{3}{2} \) and \( r = 1 - \cos \theta \) simultaneously. Eliminating \( r \), we get \( \frac{3}{2} = 1 - \cos \theta \), from which \( \theta = 2\pi/3 \) and \( 4\pi/3 \). Thus the limits are (no integrand is given):

\[
\int_0^{4\pi/3} \int_{1/\cos \theta}^{1} dr \, d\theta = \int_{3/2}^{\pi/3} \int_{1/\cos \theta}^{1} dr \, d\theta.
\]

d) The circle has polar equation \( r = 2a \cos \theta \). The line \( y = a \) has polar equation \( r \sin \theta = a \), or \( r = a \csc \theta \). Thus the limits are (no integrand):

\[
\int_0^{\pi/2} \int_{\frac{a}{2\csc \theta}}^{2a} dr \, d\theta = \frac{4\pi a^2}{3}.
\]

3B-2

a) \( \int_0^{\pi/2} \int_0^{\sin \theta} \frac{r \, d\theta \, dr}{r} = \int_0^{\pi/2} \sin 2\theta \, d\theta = \left. -\frac{1}{2} \cos 2\theta \right|_0^{\pi/2} = -\frac{1}{2}(-1 - 1) = 1.\)

b) \( \int_0^{\pi/2} \int_0^{a} r \, dr \, d\theta = \int_0^{\pi/2} \frac{\pi}{2} \cdot \frac{1}{2} \ln(1 + r^2)|_0^a = \frac{\pi}{4} \ln(1 + a^2).\)

c) \( \int_0^{\pi/4} \int_0^{\tan \theta} \tan^2 \theta \cdot r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta \, d\theta = \frac{1}{6} \tan^3 \theta \bigg|_0^{\pi/4} = \frac{1}{6}.\)

d) \( \int_0^{\pi/2} \int_0^{\sin \theta} \frac{r}{\sqrt{1 - r^2}} \, dr \, d\theta = \frac{\pi}{4} \sin \theta \bigg|_0^{\pi/2} = \frac{\pi}{4} - 1.\)

Inner: \( -\sqrt{1 - r^2} \bigg|_0^{\sin \theta} = 1 - \cos \theta \) Outer: \( \theta - \sin \theta \bigg|_0^{\pi/2} = \pi/2 - 1.\)

3B-3 a) the hemisphere is the graph of \( z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2} \), so we get

\[
\int_R \sqrt{a^2 - r^2} \, dA = \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta = 2\pi \cdot \frac{1}{3} (a^2 - r^2)^{3/2} \bigg|_0^a = \frac{2}{3} a^3 = \frac{2}{3} \pi a^3.
\]
b) \[ \int_0^{\pi/2} \int_0^a (r \cos \theta)(r \sin \theta) r \, dr \, d\theta = \int_0^a r^3 \, dr \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{a^4}{4} \cdot \frac{1}{2} = \frac{a^4}{8}. \]

c) In order to be able to use the integral formulas at the beginning of 3B, we use symmetry about the y-axis to compute the volume of just the right side, and double the answer.

\[
\int_R \sqrt{x^2 + y^2} \, dA = 2 \int_0^{\pi/2} \int_0^{2\sin \theta} r^2 \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \frac{1}{3} (2 \sin \theta)^3 \, d\theta
\]

\[= 2 \cdot \frac{8}{3} \cdot \frac{2}{3} = \frac{32}{9}, \text{ by the integral formula at the beginning of 3B.} \]

d) \[2 \int_0^{\pi/2} \int_0^{\sqrt{\cos \theta}} r^2 \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \frac{1}{4} \cos^2 \theta \, d\theta = 2 \cdot \frac{1}{4} \cdot \frac{\pi}{4} = \frac{\pi}{8}. \]

### 3C. Applications of Double Integration

**3C-1** Placing the figure so its legs are on the positive x- and y-axes,

a) M.I. = \( \int_0^a \int_0^{a-x} x^2 \, dy \, dx \) Inner: \( x^2y \bigg|_{x=a} = x^2(a-x) \); Outer: \( \frac{1}{3}x^3a - \frac{1}{3}x^4 \bigg|_0^a = \frac{1}{12}a^4 \).

b) \( \int_R (x^2 + y^2) \, dA = \int_R x^2 \, dA + \int_R y^2 \, dA = \frac{1}{12}a^4 + \frac{1}{12}a^4 = \frac{1}{6}a^4 \).

c) Divide the triangle symmetrically into two smaller triangles, their legs are \( \frac{a}{2} \);

Using the result of part (a), M.I. of \( R \) about hypotenuse = \( 2 \cdot \frac{1}{12} \left( \frac{a}{\sqrt{2}} \right)^4 \) = \( \frac{a^4}{24} \).

**3C-2** In both cases, \( \bar{x} \) is clear by symmetry; we only need \( \bar{y} \).

a) Mass is \( \int_R \pi y \, dy \, dx = \pi \)

y-moment is \( \int_R y \, dA = \int_0^\pi \int_0^{\sin x} y \, dy \, dx = \frac{1}{2} \int_0^\pi \sin^2 x \, dx = \frac{\pi}{4} \) therefore \( \bar{y} = \frac{\pi}{8} \).

b) Mass is \( \int_R y \, dA = \frac{\pi}{4} \), by part (a).

Using the formulas at the beginning of 3B,

y-moment is \( \int_R y^2 \, dA = \int_0^\pi \int_0^{\sin x} y^2 \, dy \, dx = 2 \int_0^{\pi/2} \frac{\sin^3 x}{3} \, dx = 2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9} \).

Therefore \( \bar{y} = \frac{4}{9} \cdot \frac{4}{\pi} = \frac{16}{9\pi} \).
3C-3 Place the segment either horizontally or vertically, so the diameter is respectively on the x or y axis. Find the moment of half the segment and double the answer.

(a) (Horizontally, using rectangular coordinates) Note that \( a^2 - c^2 = b^2 \).
\[
\int_c^b \int_c^{\sqrt{a^2 - x^2}} y \, dy \, dx = \int_c^b \frac{1}{2} (a^2 - x^2 - c^2) \, dx = \frac{1}{2} \left[ b^3 x - \frac{x^3}{3} \right]_c^b = \frac{1}{3} b^3; \quad \text{ans: } \frac{2}{3} b^3.
\]

(b) (Vertically, using polar coordinates). Note that \( x = c \) becomes \( r = c \csc \theta \).
\[
\text{Moment} = \int_0^\alpha \int_c^{\csc \theta} r \cos \theta \cdot r \, dr \, d\theta \quad \text{Inner: } \frac{1}{2} r^3 \cos \theta \int_c^{\csc \theta} \Big|_0^a = \frac{1}{2} (a^3 \cos \theta - c^3 \sec^2 \theta)\]
\[
\text{Outer: } \frac{1}{2} \left[ a^3 \sin \theta - c^3 \tan \theta \right]_0^\alpha = \frac{1}{2} (a^3 b - c^3 b) = \frac{1}{3} b^3; \quad \text{ans: } \frac{2}{3} b^3.
\]

3C-4 Place the sector so its vertex is at the origin and its axis of symmetry lies along the positive x-axis. By symmetry, the center of mass lies on the x-axis, so we only need find \( \bar{x} \).

Since \( \delta = 1 \), the area and mass of the disc are the same: \( \pi a^2 \cdot \frac{2a}{2\pi} = a^2 \alpha \).

\[
ex\text{-moment: } 2 \int_0^\alpha r \cos \theta \cdot r \, dr \, d\theta \quad \text{Inner: } \frac{2}{3} r^3 \cos \theta \int_0^\alpha \Big|_0^\alpha = \frac{2}{3} a^3 \sin \alpha, \quad \bar{x} = \frac{\frac{2}{3} a^3 \sin \alpha}{a^2 \alpha} = \frac{2}{3} \cdot \frac{a \sin \alpha}{\alpha}.
\]

3C-5 By symmetry, we use just the upper half of the loop and double the answer. The upper half lies between \( \theta = 0 \) and \( \theta = \pi/4 \).
\[
2 \int_0^{\pi/4} \int_0^{\sqrt{2a}} r^2 \, r \, dr \, d\theta = 2 \int_0^{\pi/4} \frac{1}{4} a^4 \cos^2 2\theta \, d\theta
\]

Putting \( u = 2\theta \), the above becomes \( \frac{a^4}{2} \cdot 2 \int_0^{\pi/2} \cos^2 u \, du = \frac{a^4}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^4}{16} \).

3D. Changing Variables

3D-1 Let \( u = x - 3y, \ v = 2x + y; \ \frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{array}{cc} 1 & -3 \\ 2 & 1 \end{array} \right| = 7; \ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{7} \).
\[
\int_R \int_{2x+y}^x (u,v) \, dx \, dy = \frac{1}{7} \int_0^7 \int_1^4 u \, dv \, du
\]

Inner: \( u \ln v \big|_1^4 = u \ln 4 \); Outer: \( \frac{1}{2} u^2 \ln 4 \big|_0^7 = \frac{49 \ln 4}{2}; \ \text{Ans: } \frac{149 \ln 4}{2} = 7 \ln 2.\]
3D-2 Let \( u = x + y, \) \( v = x - y. \) Then \( \frac{\partial(u,v)}{\partial(x,y)} = 2; \) \( \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}. \)

To get the \( uv\)-equation of the bottom of the triangular region:
\[ y = 0 \Rightarrow u = x, \quad v = x \Rightarrow u = v. \]

\[
\iint_R \cos \left( \frac{x-y}{x+y} \right) \, dx \, dy = \frac{1}{2} \iint_R \cos \frac{v}{u} \, dv \, du
\]

Inner: \( u \sin \frac{v}{u} \bigg|_0^u = u \sin 1 \quad \text{Outer:} \quad \frac{1}{2} u^2 \sin 1 \bigg|_0^2 = 2 \sin 1 \quad \text{Ans:} \quad \sin 1 \]

3D-3 Let \( u = x, \) \( v = 2y; \) \( \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \)

Letting \( R \) be the elliptical region whose boundary is \( x^2 + 4y^2 = 16 \) in \( xy\)-coordinates, and \( u^2 + v^2 = 16 \) in \( uv\)-coordinates (a circular disc), we have

\[
\iint_R (16 - x^2 - 4y^2) \, dy \, dx = \frac{1}{2} \iint_R (16 - u^2 - v^2) \, dv \, du
\]

\[
= \frac{1}{2} \int_0^{2\pi} \int_0^4 (16 - r^2) \, r \, dr \, d\theta = \pi \left( \frac{16r^2}{2} - \frac{r^4}{4} \right)_0^4 = 64\pi.
\]

3D-4 Let \( u = x + y, \) \( v = 2x - 3y; \) then \( \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5; \) \( \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5} \)

We next express the boundary of the region \( R \) in \( uv\)-coordinates.

For the \( x\)-axis, we have \( y = 0, \) so \( u = x, v = 2x, \) giving \( v = 2u. \)

For the \( y\)-axis, we have \( x = 0, \) so \( u = y, v = -3y, \) giving \( v = -3u. \)

It is best to integrate first over the lines shown, \( v = c; \) this means \( v \) is held constant, i.e., we are integrating first with respect to \( u. \) This gives

\[
\iint_R (2x - 3y)^2(x + y)^2 \, dx \, dy = \int_0^4 \int_{-\sqrt{3}u/3}^{\sqrt{3}u/3} v^2 u^2 \, du \, dv
\]

Inner: \( \frac{v^3}{15} \bigg|_{-\sqrt{3}u/3}^{\sqrt{3}u/3} = \frac{v^3}{15} \left( \frac{1}{8} - \frac{1}{27} \right) \quad \text{Outer:} \quad \frac{v^6}{6 \cdot 15} \left( \frac{1}{8} + \frac{1}{27} \right)_0^4 = \frac{4^6}{6 \cdot 15} \left( \frac{1}{8} + \frac{1}{27} \right).
\]

3D-5 Let \( u = xy, \) \( v = y/x; \) in the other direction this gives \( v^2 = uv, x^2 = u/v. \)

We have \( \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} = 2v; \) \( \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}; \) this gives

\[
\iint_R (x^2 + y^2) \, dx \, dy = \int_0^3 \int_0^1 \frac{1}{2v} (u + uv) \, dv \, du.
\]

Inner: \( \frac{-u}{2v} + \frac{u}{2} \bigg|_1^0 = u \left( \frac{-1}{4} + 1 - \frac{1}{2} \right) = \frac{3u}{4}; \) Outer: \( \frac{3u^2}{8} \bigg|_0^3 = \frac{27}{8}. \)

3D-8 a) \( y = x^2; \) therefore \( u = x^3, v = x, \) which gives \( u = v^3. \)

b) We get \( \frac{u}{v} + uv = 1, \) or \( u = -\frac{v}{v^2 + 1}; \) (cf. 3D-5)