1. Vectors and Matrices

1A. Vectors

**Definition.** A direction is just a unit vector. The direction of \( \mathbf{A} \) is defined by

\[
\text{dir } \mathbf{A} = \frac{\mathbf{A}}{|\mathbf{A}|}, \quad (\mathbf{A} \neq 0);
\]

it is the unit vector lying along \( \mathbf{A} \) and pointed like \( \mathbf{A} \) (not like \( -\mathbf{A} \)).

1A-1 Find the magnitude and direction (see the definition above) of the vectors

a) \( \mathbf{i} + \mathbf{j} + \mathbf{k} \)  
b) \( 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \)  
c) \( 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k} \)

1A-2 For what value(s) of \( c \) will \( \mathbf{i} - \mathbf{j} + c\mathbf{k} \) be a unit vector?

1A-3 a) If \( P = (1, 3, -1) \) and \( Q = (0, 1, 1) \), find \( \mathbf{A} = PQ, \ |\mathbf{A}|, \text{ and } \text{dir } \mathbf{A} \).

b) A vector \( \mathbf{A} \) has magnitude 6 and direction \( (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})/3 \). If its tail is at \((-2, 0, 1)\), where is its head?

1A-4 a) Let \( P \) and \( Q \) be two points in space, and \( X \) the midpoint of the line segment \( PQ \). Let \( O \) be an arbitrary fixed point; show that as vectors, \( O\mathbf{X} = \frac{1}{2}(OP + OQ) \).

b) With the notation of part (a), assume that \( X \) divides the line segment \( PQ \) in the ratio \( r:s \), where \( r+s = 1 \). Derive an expression for \( O\mathbf{X} \) in terms of \( OP \) and \( OQ \).

1A-5 What are the \( i \ j \)-components of a plane vector \( \mathbf{A} \) of length 3, if it makes an angle of 30° with \( \mathbf{i} \) and 60° with \( \mathbf{j} \). Is the second condition redundant?

1A-6 A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the \( i \ j \)-components).

1A-7 Let \( \mathbf{A} = ai + bj \) be a plane vector; find in terms of \( a \) and \( b \) the vectors \( \mathbf{A}' \) and \( \mathbf{A}'' \) resulting from rotating \( \mathbf{A} \) by 90° a) clockwise  
b) counterclockwise.

(Hint: make \( \mathbf{A} \) the diagonal of a rectangle with sides on the \( x \) and \( y \)-axes, and rotate the whole rectangle.)

c) Let \( \mathbf{i}' = (3\mathbf{i} + 4\mathbf{j})/5 \). Show that \( \mathbf{i}' \) is a unit vector, and use the first part of the exercise to find a vector \( \mathbf{j}' \) such that \( \mathbf{i}', \mathbf{j}' \) forms a right-handed coordinate system.

1A-8 The direction (see definition above) of a space vector is in engineering practice often given by its **direction cosines**. To describe these, let \( \mathbf{A} = ai + bj + ck \) be a space vector, represented as an origin vector, and let \( \alpha, \beta, \) and \( \gamma \) be the three angles \( (\leq \pi) \) that \( \mathbf{A} \) makes respectively with \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \).

a) Show that \( \text{dir } \mathbf{A} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k} \). (The three coefficients are called the **direction cosines** of \( \mathbf{A} \).)

b) Express the direction cosines of \( \mathbf{A} \) in terms of \( a, b, c \); find the direction cosines of the vector \(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\).

c) Prove that three numbers \( t, u, v \) are the direction cosines of a vector in space if and only if they satisfy \( t^2 + u^2 + v^2 = 1 \).
1A-9 Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides A and B.)

1A-10 Prove using vector methods (without components) that the midpoints of the sides of a space quadrilateral form a parallelogram.

1A-11 Prove using vector methods (without components) that the diagonals of a parallelogram bisect each other. (One way: let X and Y be the midpoints of the two diagonals; show \( X = Y \).)

1A-12* Label the four vertices of a parallelogram in counterclockwise order as OPQR. Prove that the line segment from O to the midpoint of PQ intersects the diagonal PR in a point X that is 1/3 of the way from P to R.

Let \( A = OP \), and \( B = OR \); express everything in terms of \( A \) and \( B \).

1A-13* a) Take a triangle \( PQR \) in the plane; prove that as vectors \( PQ + QR + RP = 0 \).

b) Continuing part a), let \( A \) be a vector the same length as \( PQ \), but perpendicular to it, and pointing outside the triangle. Using similar vectors \( B \) and \( C \) for the other two sides, prove that \( A + B + C = 0 \). (This only takes one sentence, and no computation.)

1A-14* Generalize parts a) and b) of the previous exercise to a closed polygon in the plane which doesn’t cross itself (i.e., one whose interior is a single region); label its vertices \( P_1, P_2, \ldots, P_n \) as you walk around it.

1A-15* Let \( P_1, \ldots, P_n \) be the vertices of a regular \( n \)-gon in the plane, and \( O \) its center; show without computation or coordinates that \( OP_1 + OP_2 + \ldots + OP_n = 0 \),

a) if \( n \) is even; b) if \( n \) is odd.

1B. Dot Product

1B-1 Find the angle between the vectors

\[
\text{a) } i - k \quad \text{and} \quad 4i + 4j - 2k \quad \text{b) } i + j + 2k \quad \text{and} \quad 2i - j + k.
\]

1B-2 Tell for what values of \( c \) the vectors \( ci + 2j - k \) and \( i - j + 2k \) will

a) be orthogonal \quad b) form an acute angle

1B-3 Using vectors, find the angle between a longest diagonal \( PQ \) of a cube, and

a) a diagonal \( PR \) of one of its faces; \quad b) an edge \( PS \) of the cube.

(Choose a size and position for the cube that makes calculation easiest.)

1B-4 Three points in space are \( P : (a, 1, -1) \), \( Q : (0, 1, 1) \), \( R : (a, -1, 3) \). For what value(s) of \( a \) will \( PQR \) be

a) a right angle \quad b) an acute angle?

1B-5 Find the component of the force \( F = 2i - 2j + k \) in

a) the direction \( \frac{i + j - k}{\sqrt{3}} \) \quad b) the direction of the vector \( 3i + 2j - 6k \).
1B-6 Let \( O \) be the origin, \( c \) a given number, and \( u \) a given direction (i.e., a unit vector). Describe geometrically the locus of all points \( P \) in space that satisfy the vector equation

\[ \mathbf{OP} \cdot \mathbf{u} = c |\mathbf{OP}|. \]

In particular, tell for what value(s) of \( c \) the locus will be

a) a plane  b) a ray (i.e., a half-line)  c) empty

(Hint: divide through by \(|\mathbf{OP}|\).)

1B-7 a) Verify that \( \mathbf{i}' = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} \) and \( \mathbf{j}' = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}} \) are perpendicular unit vectors that form a right-handed coordinate system.

b) Express the vector \( \mathbf{A} = 2\mathbf{i} - 3\mathbf{j} \) in the \( \mathbf{i}'\mathbf{j}' \)-system by using the dot product.

c) Do b) a different way, by solving for \( \mathbf{i} \) and \( \mathbf{j} \) in terms of \( \mathbf{i}' \) and \( \mathbf{j}' \) and then substituting into the expression for \( \mathbf{A} \).

1B-8 The vectors \( \mathbf{i}' = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} \), \( \mathbf{j}' = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} \), and \( \mathbf{k}' = \frac{\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{6}} \) are three mutually perpendicular unit vectors that form a right-handed coordinate system.

a) Verify this.  b) Express \( \mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \) in this system (cf. 1B-7b)

1B-9 Let \( \mathbf{A} \) and \( \mathbf{B} \) be two plane vectors, neither one of which is a multiple of the other. Express \( \mathbf{B} \) as the sum of two vectors, one a multiple of \( \mathbf{A} \), and the other perpendicular to \( \mathbf{A} \); give the answer in terms of \( \mathbf{A} \) and \( \mathbf{B} \).

(Hint: let \( \mathbf{u} = \text{dir } \mathbf{A} \); what's the \( \mathbf{u} \)-component of \( \mathbf{B} \)?)

1B-10 Prove using vector methods (without components) that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

1B-11 Prove using vector methods (without components) that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides have equal lengths.

1B-12 Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

1B-13 Prove the trigonometric formula: \( \cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \).

(Hint: consider two unit vectors making angles \( \theta_1 \) and \( \theta_2 \) with the positive \( x \)-axis.)

1B-14 Prove the law of cosines: \( c^2 = a^2 + b^2 - 2ab \cos \theta \) by using the algebraic laws for the dot product and its geometric interpretation.

1B-15* The Cauchy-Schwarz inequality

a) Prove from the geometric definition of the dot product the following inequality for vectors in the plane or in space:

\[ |\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|. \]

Under what circumstances does equality hold?

b) If the vectors are plane vectors, write out what this inequality says in terms of \( \mathbf{i} \mathbf{j} \)-components.
c) Give a different argument for the inequality (*) as follows (this argument generalizes to $n$-dimensional space):

i) for all values of $t$, we have $(A + tB) \cdot (A + tB) \geq 0$;

ii) use the algebraic laws of the dot product to write the expression in (i) as a quadratic polynomial in $t$;

iii) by (i) this polynomial has at most one zero; this implies by the quadratic formula that its coefficients must satisfy a certain inequality — what is it?

1C. Determinants

1C-1 Calculate the value of the determinants  

a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$  

b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}$

1C-2 Calculate $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$ using the Laplace expansion by the cofactors of:

a) the first row  
b) the first column

1C-3 Find the area of the plane triangle whose vertices lie at  

a) $(0, 0), (1, 2), (1, -1)$;  
b) $(1, 2), (1, -1), (2, 3)$.

1C-4 Show that $\begin{vmatrix} x_1 & x_2 & 1 \\ x_2 & x_3 & 1 \\ x_1 & x_3 & 1 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$.

(This type of determinant is called a Vandermonde determinant.)

1C-5 a) Show that the value of a $2 \times 2$ determinant is unchanged if you add to the second row a scalar multiple of the first row.

b) Same question, with “row” replaced by “column”.

1C-6 Use a Laplace expansion and Exercise 5a to show the value of a $3 \times 3$ determinant is unchanged if you add to the second row a scalar multiple of the third row.

1C-7 Let $(x_1, y_1)$ and $(x_2, y_2)$ be two unit vectors. Find the maximum value of the function $f(x_1, x_2, y_1, y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

1C-8* The base of a parallelepiped is a parallelogram whose edges are the vectors $b$ and $c$, while its third edge is the vector $a$. (All three vectors have their tail at the same vertex; one calls them “coterminal”.)

a) Show that the volume of the parallelepiped $abc$ is $\pm a \cdot (b \times c)$.

b) Show that $a \cdot (b \times c) = \text{the determinant whose rows are respectively the components of the vectors } a, b, c$.

(These two parts prove (3), the volume interpretation of a $3 \times 3$ determinant.)
1C-9 Use the formula in Exercise 1C-8 to calculate the volume of a tetrahedron having as vertices $(0,0,0), (0,-1,2), (0,1,-1), (1,2,1)$. (The volume of a tetrahedron is $\frac{1}{3}$ (base)(height).)

1C-10 Show by using Exercise 8 that if three origin vectors lie in the same plane, the determinant having the three vectors as its three rows has the value zero.

1D. Cross Product

1D-1 Find $A \times B$ if 
   a) $A = i - 2j + k, \quad B = 2i - j - k$
   b) $A = 2i - 3k, \quad B = i + j - k$

1D-2 Find the area of the triangle in space having its vertices at the points
   $P : (2,0,1), \quad Q : (3,1,0), \quad R : (-1,1,-1)$.

1D-3 Two vectors $i$ and $j$ of a right-handed coordinate system are to have the directions respectively of the vectors $A = 2i - j$ and $B = i + 2j + k$. Find all three vectors $i', j', k'$.

1D-4 Verify that the cross product $\times$ does not in general satisfy the associative law, by showing that for the particular vectors $i, i, j$, we have $(i \times i) \times j \neq i \times (i \times j)$.

1D-5 What can you conclude about $A$ and $B$
   a) if $|A \times B| = |A||B|$
   b) if $|A \times B| = A \cdot B$.

1D-6 Take three faces of a unit cube having a common vertex $P$; each face has a diagonal ending at $P$; what is the volume of the parallelepiped having these three diagonals as coterminal edges?

1D-7 Find the volume of the tetrahedron having vertices at the four points
   $P : (1,0,1), \quad Q : (-1,1,2), \quad R : (0,0,2), \quad S : (3,1,-1)$.
   Hint: volume of tetrahedron $= \frac{1}{6}$ (volume of parallelepiped with same 3 coterminal edges)

1D-8 Prove that $A \cdot (B \times C) = (A \times B) \cdot C$, by using the determinantal formula for the scalar triple product, and the algebraic laws of determinants in Notes D.

1D-9 Show that the area of a triangle in the $xy$-plane having vertices at $(x_i, y_i)$, for $i = 1, 2, 3$, is given by the determinant
   $$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$
   Do this two ways:
   a) by relating the area of the triangle to the volume of a certain parallelepiped
   b) by using the laws of determinants (p. L.1 of the notes) to relate this determinant to the $2 \times 2$ determinant that would normally be used to calculate the area.
1E. Equations of Lines and Planes

1E-1 Find the equations of the following planes:
   a) through (2,0,-1) and perpendicular to \( i + 2j - 2k \)
   b) through the origin, (1,1,0), and (2,-1,3)
   c) through (1,0,1), (2,-1,2), (-1,3,2)
   d) through the points on the x, y and z-axes where \( x = a, y = b, z = c \) respectively
      (give the equation in the form \( Ax + By + Cz = 1 \) and remember it)
   e) through (1,0,1) and (0,1,1) and parallel to \( i - j + 2k \)

1E-2 Find the dihedral angle between the planes \( 2x - y + z = 3 \) and \( x + y + 2z = 1 \).

1E-3 Find in parametric form the equations for
   a) the line through (1,0,-1) and parallel to \( 2i - j + 3k \)
   b) the line through (2,-1,-1) and perpendicular to the plane \( x - y + 2z = 3 \)
   c) all lines passing through (1,1,1) and lying in the plane \( x + 2y - z = 2 \)

1E-4 Where does the line through (0,1,2) and (2,0,3) intersect the plane \( x + 4y + z = 4 \)?

1E-5 The line passing through (1,1,-1) and perpendicular to the plane \( x + 2y - z = 3 \) intersects the plane \( 2x - y + z = 1 \) at what point?

1E-6 Show that the distance \( D \) from the origin to the plane \( ax + by + cz = d \) is given by
   \[ D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}. \]
   (Hint: Let \( \mathbf{n} \) be the unit normal to the plane. and \( P \) be a point on the plane; consider the component of \( OP \) in the direction \( \mathbf{n} \)).

1E-7* Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).
   (Hint: the shortest line segment joining the two skew lines will be perpendicular to both of them (if it weren’t, it could be shortened.).)

1F. Matrix Algebra

1F-1* Let \( A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 2 \\ -3 & 4 \end{pmatrix} \). Compute
   a) \( B + C, \ B - C, \ 2B - 3C \).
   b) \( AB, AC, BA, CA, BC^T, CB^T \)
   c) \( A(B + C), AB + AC; (B + C)A, BA + CA \)

1F-2* Let \( A \) be an arbitrary \( m \times n \) matrix, and let \( I_k \) be the identity matrix of size \( k \). Verify that \( I_mA = A \) and \( AI_n = A \).

1F-3 Find all \( 2 \times 2 \) matrices \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).
1F-4* Show that matrix multiplication is not in general commutative by calculating for each pair below the matrix $AB - BA$:

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}$

1F-5 a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute $A^2, A^3, A^n$. b) Do the same for $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1F-6* Let $A, A', B, B'$ be $2 \times 2$ matrices, and $O$ the $2 \times 2$ zero matrix. Express in terms of these five matrices the product of the $4 \times 4$ matrices $\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} A' & O \\ O & B' \end{pmatrix}$.

1F-7* Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Show there are no values of $a$ and $b$ such that $AB - BA = I_2$.

1F-8 a) If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, what is the $3 \times 3$ matrix $A$?

b) If $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, what is the matrix $A$?

1F-9 A square $n \times n$ matrix is called **orthogonal** if $A \cdot A^T = I_n$. Show that this condition is equivalent to saying that

a) each row of $A$ is a row vector of length 1,

b) two different rows are orthogonal vectors.

1F-10* Suppose $A$ is a $2 \times 2$ orthogonal matrix, whose first entry is $a_{11} = \cos \theta$. Fill in the rest of $A$. (There are four possibilities. Use Exercise 9.)

1F-11* Show that if $A + B$ and $AB$ are defined, then

a) $(A + B)^T = A^T + B^T$,  
   b) $(AB)^T = B^T A^T$.

1G. Solving Square Systems; Inverse Matrices

For each of the following, solve the equation $Ax = b$ by finding $A^{-1}$.

1G-1* $A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$.

1G-2* a) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$;  
   b) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

1G-3 $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. Solve $Ax = b$ by finding $A^{-1}$. 

1G-4 Referring to Exercise 3 above, solve the system

\[\begin{align*}
x_1 - x_2 + x_3 &= y_1, \\
x_2 + x_3 &= y_2, \\
-x_1 - x_2 + 2x_3 &= y_3
\end{align*}\]

for the \(x_i\) as functions of the \(y_i\).

1G-5 Show that \((AB)^{-1} = B^{-1}A^{-1}\), by using the definition of inverse matrix.

1G-6* Another calculation of the inverse matrix.

If we know \(A^{-1}\), we can solve the system \(Ax = y\) for \(x\) by writing \(x = A^{-1}y\). But conversely, if we can solve by some other method (elimination, say) for \(x\) in terms of \(y\), getting \(x = By\), then the matrix \(B = A^{-1}\), and we will have found \(A^{-1}\).

This is a good method if \(A\) is an upper or lower triangular matrix — one with only zeros respectively below or above the main diagonal. To illustrate:

\[A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -x_1 + x_2 + 3x_3 &= y_1 \end{pmatrix}\]

a) Let \(A\) = \[\begin{pmatrix} -1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}\]; find \(A^{-1}\) by solving \(2x_2 - x_3 = y_2\) for the \(x_i\) in terms of the \(y_i\) (start from the bottom and proceed upwards).

b) Calculate \(A^{-1}\) by the method given in the notes.

1G-7* Consider the rotation matrix \(A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\) corresponding to rotation of the \(x\) and \(y\) axes through the angle \(\theta\). Calculate \(A_\theta^{-1}\) by the adjoint matrix method, and explain why your answer looks the way it does.

1G-8* a) Show: \(A\) is an orthogonal matrix (cf. Exercise 1F-9) if and only if \(A^{-1} = A^T\).

b) Illustrate with the matrix of exercise 7 above.

c) Use (a) to show that if \(A\) and \(B\) are \(n \times n\) orthogonal matrices, so is \(AB\).

1G-9* a) Let \(A\) be a \(3 \times 3\) matrix such that \(|A| \neq 0\). The notes construct a right-inverse \(A^{-1}\), that is, a matrix such that \(A \cdot A^{-1} = I\). Show that every such matrix \(A\) also has a left inverse \(B\) (i.e., a matrix such that \(BA = I\)).

(Hint: Consider the equation \(A^T(A^T)^{-1} = I\); cf. Exercise 1F-11.)

b) Deduce that \(B = A^{-1}\) by a one-line argument.

(This shows that the right inverse \(A^{-1}\) is automatically the left inverse also. So if you want to check that two matrices are inverses, you only have to do the multiplication on one side — the product in the other order will automatically be \(I\) also.)

1G-10* Let \(A\) and \(B\) be two \(n \times n\) matrices. Suppose that \(B = P^{-1}AP\) for some invertible \(n \times n\) matrix \(P\). Show that \(B^n = P^{-1}A^nP\). If \(B = I_n\), what is \(A\)?

1G-11* Repeat Exercise 6a and 6b above, doing it this time for the general \(2 \times 2\) matrix \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), assuming \(|A| \neq 0\).
1H. Cramer's Rule; Theorems about Square Systems

1H-1 Use Cramer's rule to solve for \( x \) in the following:

\[
\begin{align*}
3x - y + z &= 1, & x - y + z &= 0 \\
-x + 2y + z &= 2, & x - z &= 1. \\
x - y + z &= -3, & -x + y + z &= 2
\end{align*}
\]

1H-2 Using Cramer's rule, give another proof that if \( A \) is an \( n \times n \) matrix whose determinant is non-zero, then the equations \( Ax = 0 \) have only the trivial solution.

\[
x_1 - x_2 + x_3 = 0
\]

1H-3 a) For what \( c \)-value(s) will \( 2x_1 + x_2 + x_3 = 0 \) have a non-trivial solution?

\[-x_1 + cx_2 + 2x_3 = 0
\]

b) For what \( c \)-value(s) will \( \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix} \) have a non-trivial solution? (Write it as a system of homogeneous equations.)

c) For each value of \( c \) in part (a), find a non-trivial solution to the corresponding system. (Interpret the equations as asking for a vector orthogonal to three given vectors; find it by using the cross product.)

d)* For each value of \( c \) in part (b), find a non-trivial solution to the corresponding system.

\[x - 2y + z = 0
\]

1H-4* Find all solutions to the homogeneous system

\[
\begin{align*}
x + y - z &= 0; \\
3x - 3x + z &= 0
\end{align*}
\]

use the method suggested in Exercise 3c above.

1H-5 Suppose that for the system

\[
\begin{align*}
a_1x + b_1y &= c_1 \\
a_2x + b_2y &= c_2
\end{align*}
\]

we have \( \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \). Assume that \( a_1 \neq 0 \). Show that the system is consistent (i.e., has solutions) if and only if \( c_2 = \frac{a_2}{a_1}c_1 \).

1H-6* Suppose \( |A| = 0 \), and that \( x_1 \) is a particular solution of the system \( Ax = B \). Show that any other solution \( x_2 \) of this system can be written as \( x_2 = x_1 + x_0 \), where \( x_0 \) is a solution of the system \( Ax = 0 \).

1H-7 Suppose we want to find a pure oscillation (sine wave) of frequency 1 passing through two given points. In other words, we want to choose constants \( a \) and \( b \) so that the function

\[f(x) = a \cos x + b \sin x
\]

has prescribed values at two given \( x \)-values: \( f(x_1) = y_1, \ f(x_2) = y_2 \).

a) Show this is possible in one and only one way, if we assume that \( x_2 \neq x_1 + n\pi \), for every integer \( n \).

b) If \( x_2 = x_1 + n\pi \) for some integer \( n \), when can \( a \) and \( b \) be found?
1H-8* The method of partial fractions, if you do it by undetermined coefficients, leads to a system of linear equations. Consider the simplest case:

\[
\frac{ax + b}{x - r_1}(x - r_2) = \frac{c}{x - r_1} + \frac{d}{x - r_2}, \quad (a, b \text{ given}; c, d \text{ to be found});
\]

what are the linear equations which determine the constants \(c\) and \(d\)? Under what circumstances do they have a unique solution?

(If you are ambitious, try doing this also for three roots \(r_i, \ i = 1, 2, 3\). Evaluate the determinant by using column operations to get zeros in the top row.)

### 11. Vector Functions and Parametric Equations

11-1 The point \(P\) moves with constant speed \(v\) in the direction of the constant vector \(a\mathbf{i} + b\mathbf{j}\). If at time \(t = 0\) it is at \((x_0, y_0)\), what is its position vector function \(r(t)\)?

11-2 A point moves \textit{clockwise} with constant angular velocity \(\omega\) on the circle of radius \(a\) centered at the origin. What is its position vector function \(r(t)\), if at time \(t = 0\) it is at \(a) \ (a, 0) \quad b) \ (0, a)\)

11-3 Describe the motions given by each of the following position vector functions, as \(t\) goes from \(-\infty\) to \(\infty\). In each case, give the \(xy\)-equation of the curve along which \(P\) travels, and tell what part of the curve is actually traced out by \(P\).

\[
\begin{align*}
\text{a) } r &= 2\cos^2 t \mathbf{i} + \sin^2 t \mathbf{j} \\
\text{b) } r &= \cos 2t \mathbf{i} + \cos t \mathbf{j} \\
\text{c) } r &= (t^2 + 1) \mathbf{i} + t^3 \mathbf{j} \\
\text{d) } r &= \tan t \mathbf{i} + \sec t \mathbf{j}
\end{align*}
\]

11-4 A roll of plastic tape of outer radius \(a\) is held in a fixed position while the tape is being unwound counterclockwise. The end \(P\) of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin \(O\), and the end \(P\) to be initially at \((a, 0)\), write parametric equations for the motion of \(P\).

(Use vectors; express the position vector \(OP\) as a vector function of one variable.)

11-5 A string is wound clockwise around the circle of radius \(a\) centered at the origin \(O\); the initial position of the end \(P\) of the string is \((a, 0)\). Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of \(P\).

(Use vectors; express the position vector \(OP\) as a vector function of one variable.)

11-6 A bow-and-arrow hunter walks toward the origin along the positive \(x\)-axis, with unit speed; at time 0 he is at \(x = 10\). His arrow (of unit length) is aimed always toward a rabbit hopping with constant velocity \(\sqrt{5}\) in the first quadrant along the line \(y = 2x\); at time 0 it is at the origin.

\begin{enumerate}
\item Write down the vector function \(\mathbf{A}(t)\) for the arrow at time \(t\).
\item The hunter shoots (and misses) when closest to the rabbit; when is that?
\end{enumerate}

11-7 The cycloid is the curve traced out by a fixed point \(P\) on a circle of radius \(a\) which rolls along the \(x\)-axis in the positive direction, starting when \(P\) is at the origin \(O\). Find the vector function \(OP\); use as variable the angle \(\theta\) through which the circle has rolled.

(Hint: begin by expressing \(OP\) as the sum of three simpler vector functions.)
1J. Differentiation of Vector Functions

1J-1 1. For each of the following vector functions of time, calculate the velocity, speed \( |ds/dt| \), unit tangent vector (in the direction of velocity), and acceleration.
   a) \( e^t \mathbf{i} + e^{-t} \mathbf{j} \)
   b) \( t^2 \mathbf{i} + t^3 \mathbf{j} \)
   c) \( (1 - 2t^2) \mathbf{i} + t^2 \mathbf{j} + (-2 + 2t^2) \mathbf{k} \)

1J-2 Let \( \mathbf{OP} = \frac{1}{1 + t^2} \mathbf{i} + \frac{t}{1 + t^2} \mathbf{j} \) be the position vector for a motion.
   a) Calculate \( \mathbf{v} \), \( |ds/dt| \), and \( \mathbf{T} \).
   b) At what point in the speed greatest? smallest?
   c) Find the \( xy \)-equation of the curve along which the point \( P \) is moving, and describe it geometrically.

1J-3 Prove the rule for differentiating the scalar product of two plane vector functions:
\[
\frac{d}{dt} \mathbf{r} \cdot \mathbf{s} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \frac{d\mathbf{s}}{dt},
\]
by calculating with components, letting \( \mathbf{r} = x_1 \mathbf{i} + y_1 \mathbf{j} \) and \( \mathbf{s} = x_2 \mathbf{i} + y_2 \mathbf{j} \).

1J-4 Suppose a point \( P \) moves on the surface of a sphere with center at the origin; let \( \mathbf{OP} = \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \).
   Show that the velocity vector \( \mathbf{v} \) is always perpendicular to \( \mathbf{r} \) two different ways:
   a) using the \( x, y, z \)-coordinates
   b) without coordinates (use the formula in 1J-3, which is valid also in space).
   c) Prove the converse: if \( \mathbf{r} \) and \( \mathbf{v} \) are perpendicular, then the motion of \( P \) is on the surface of a sphere.

1J-5 a) Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular. (Use the formula in 1J-3.)
   b) Show the converse: if the velocity and acceleration vectors are perpendicular, the point \( P \) moves with constant speed.

1J-6 For the helical motion \( \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k} \),
   a) calculate \( \mathbf{v}, \mathbf{a}, \mathbf{T}, |ds/dt| \)
   b) show that \( \mathbf{v} \) and \( \mathbf{a} \) are perpendicular; explain using 1J-5

1J-7 a) Suppose you have a differentiable vector function \( \mathbf{r}(t) \). How can you tell if the parameter \( t \) is the arclength \( s \) (measured from some point in the direction of increasing \( t \)) without actually having to calculate \( s \) explicitly?
   b) How should \( a \) be chosen so that \( t \) is the arclength if \( \mathbf{r}(t) = (x_0 + at) \mathbf{i} + (y_0 + at) \mathbf{j} \)?
   c) How should \( a \) and \( b \) be chosen so that \( t \) is the arclength in the helical motion described in Exercise 1J-6?
1J-8  a) Prove the formula \( \frac{d}{dt} u(t) \mathbf{r}(t) = \frac{du}{dt} \mathbf{r}(t) + u(t) \frac{d\mathbf{r}}{dt} \).

(You may assume the vectors are in the plane; calculate with the components.)

b) Let \( \mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} \), the exponential spiral. Use part (a) to find the speed of this motion.

1J-9  A point \( P \) is moving in space, with position vector

\[ \mathbf{r} = OP = 3 \cos t \mathbf{i} + 5 \sin t \mathbf{j} + 4 \cos t \mathbf{k} \]

a) Show it moves on the surface of a sphere.

b) Show its speed is constant.

c) Show the acceleration is directed toward the origin.

d) Show it moves in a plane through the origin.

e) Describe the path of the point.

1J-10  The positive curvature \( \kappa \) of the vector function \( \mathbf{r}(t) \) is defined by \( \kappa = \left| \frac{dT}{ds} \right| \).

a) Show that the helix of 1J-6 has constant curvature. (It is not necessary to calculate \( s \) explicitly; calculate \( dT/dt \) instead and relate it to \( \kappa \) by using the chain rule.)

b) What is this curvature if the helix is reduced to a circle in the xy-plane?

1K. Kepler’s Second Law

1K-1  Prove the rule (1) in Notes K for differentiating the dot product of two plane vectors: do the calculation using an \( \mathbf{i} \mathbf{j} \)-coordinate system.

(Let \( \mathbf{r}(t) = x_1(t) \mathbf{i} + y_1(t) \mathbf{j} \) and \( \mathbf{s}(t) = x_2(t) \mathbf{i} + y_2(t) \mathbf{j} \).)

1K-2  Let \( \mathbf{s}(t) \) be a vector function. Prove by using components that

\[ \frac{ds}{dt} = 0 \quad \Rightarrow \quad \mathbf{s}(t) = \mathbf{K}, \] where \( \mathbf{K} \) is a constant vector.

1K-3  In Notes K, by reversing the steps (5) – (8), prove the statement in the last paragraph. You will need the statement in exercise 1K-2.