Problem 1

(20) Find the mass of the solid cylinder $0 \leq x^2 + y^2 \leq a^2$, $0 \leq z \leq b$, with density $\delta(x, y, z) = x^2 z$. 
Problem 2

(15) Express the average value of $z^{10}$ on the surface of the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$, as an integral in spherical coordinates. (Do not evaluate.)
Problem 3

a) (5) Explain why $\mathbf{F} = (y, x + az, y + 1)$ cannot be a gradient field unless $a = 1$.

b) (10) Next, let $a = 1$. Then $\mathbf{F} = (y, x + z, y + 1) = \nabla(xy + yz + z)$.

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $C$ given by $x = \cos^3 t$, $y = t$, $z = \sin^3 t$, $0 \leq t \leq \pi$. 
Problem 4

Consider \( \mathbf{F} = \mathbf{i} \) and \( D \) the solid quarter of a ball given by \( x^2 + y^2 + z^2 < 1, x < 0 \) and \( z > 0 \). Let \( S = S_1 + S_2 + S_3 \) denote the surface that encloses \( D \), with \( S_1 \) the flat face in the \( xy \)-plane, \( S_2 \) the flat face in the \( yz \)-plane, and \( S_3 \) the curved face.

a) (15) State the divergence theorem, and use it to find the flux out of the curved face from the fluxes through the flat faces.

b) (10) Find the integrand \( f(x, y) \) in the integral formula for the flux you found indirectly in part (a), that is,

\[
\text{flux of } \mathbf{F} \text{ out of } S_3 = \iint_{x^2 + y^2 < 1, \ z < 0} f(x, y) \, dx \, dy
\]

Do not evaluate the integral, and do not calculate the limits of integration.
(The region of integration is the projection (shadow) of \( S_3 \) in the \( xy \)-plane.)
Problem 5

(25) Consider the surface $S$ which is the portion of the plane $2y + z = 0$ in the cylinder $x^2 + y^2 \leq 1$. Its boundary curve $C$ is the ellipse given by $x^2 + y^2 = 1$, $z = -2y$.

State Stokes' theorem, and confirm it by direct computation for $\mathbf{F} = z\mathbf{i}$ on $S$. 
Problem 6 – Extra credit (10 points)

(10) Let \( F = y\mathbf{i} + 2z\mathbf{j} \). Suppose that \( \int_C F \cdot dr = 0 \) for every curve in the plane \( a\mathbf{x} + by + cz = d \).

What can be said about \( a, b, c, \) and \( d \)?