Problem 1. a) Draw a picture of the region of integration of \( \int_0^1 \int_x^{2x} dy \, dx \)

b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order \( dx \, dy \). Warning: your answer will have two pieces.

Problem 2. a) Find the mass \( M \) of the upper half of the annulus, 
\[ 1 < x^2 + y^2 < 9 \quad (y \geq 0) \] with density \( \delta = \frac{y}{x^2 + y^2} \).

b) Express the \( x \)-coordinate of the center of mass, \( \bar{x} \), as an iterated integral. (Write explicitly the integrand and limits of integration.) Without evaluating the integral, explain why \( \bar{x} = 0 \).

Problem 3. a) Show that \( F = (3x^2 - 6y^2)i + (-12xy + 4y)j \) is conservative.

b) Find a potential function for \( F \).

c) Let \( C \) be the curve \( x = 1 + y^3(1 - y)^3, \) \( 0 \leq y \leq 1 \). Calculate \( \int_C F \cdot dr \).

Problem 4. a) Express the work done by the force field \( F = (5x + 3y)i + (1 + \cos y)j \) on a particle moving counterclockwise once around the unit circle centered at the origin in the form \( \int_a^b f(t) \, dt \). (Do not evaluate the integral; don’t even simplify \( f(t) \).)

b) Evaluate the line integral using Green’s theorem.

Problem 5. Consider the rectangle \( R \) with vertices \((0,0), (1,0), (1,4)\) and \((0,4)\). The boundary of \( R \) is the curve \( C \), consisting of \( C_1 \), the segment from \((0,0)\) to \((1,0)\), \( C_2 \), the segment from \((1,0)\) to \((1,4)\), \( C_3 \) the segment from \((1,4)\) to \((0,4)\) and \( C_4 \) the segment from \((0,4)\) to \((0,0)\). Consider the vector field
\[ F = (\cos x \sin y)i + (xy + \sin x \cos y)j \]
a) Find the work of \( F \) along the boundary \( C \) oriented in a counterclockwise direction.

b) Is the total work along \( C_1, C_2 \) and \( C_3 \), more than, less than or equal to the work along \( C_4 \)?

Problem 6. Find the volume of the region enclosed by the plane \( z = 4 \) and the surface
\[ z = (2x - y)^2 + (x + y - 1)^2. \]
Suggestion: change of variables.
1. a) \[ x = 1, \quad y = 2 \] 

b) \[ \int_{y/2}^{y} dx dy + \int_{y/2}^{1} dx dy. \]

2. a) \[ \delta dA = \frac{r \sin \theta}{r^2} r \, dr \, d\theta = \sin \theta \, dr \, d\theta. \]

\[ M = \int_{R} \int_{R} \delta dA = \int_{0}^{\pi} \int_{1}^{3} \sin \theta \, d\theta \, dr = \int_{0}^{\pi} 2 \sin \theta \, d\theta = [ -2 \cos \theta ]_{0}^{\pi} = 4. \]

b) \[ \bar{x} = \frac{1}{M} \int_{R} \int_{R} x \delta dA = \frac{1}{4} \int_{0}^{\pi} \int_{1}^{3} r \cos \theta \, d\theta \, dr \]

The reason why one knows that \( \bar{x} = 0 \) without computation is that \( x \) is odd with respect to the \( y \)-axis whereas the region and the density are symmetric with respect to the \( y \)-axis: \( (x, y) \to (-x, y) \) preserves the half annulus and \( \delta(x, y) = \delta(-x, y) \).

3. a) \( N_x = -12y = M_y \), hence \( \mathbf{F} \) is conservative.

b) \( f_x = 3x^2 - 6y^2 \Rightarrow f = x^3 - 6y^2x + c(y) \Rightarrow f_y = -12xy + c'(y) = -12xy + 4y \). So \( c'(y) = 4y \), thus \( c(y) = 2y^2 \) (constant). In conclusion \( f = x^3 - 6xy^2 + 2y^2 \) (constant).

c) The curve \( C \) starts at \( (1, 0) \) and ends at \( (1, 1) \), therefore 
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(1, 0) = (1 - 6 + 2) - 1 = -4. \]

4. a) The parametrization of the circle \( C \) is \( x = \cos t, \quad y = \sin t \), for \( 0 \leq t < 2\pi \); then \( dx = -\sin t \, dt, \quad dy = \cos t \, dt \) and 
\[ W = \int_{0}^{2\pi} (5 \cos t + 3 \sin t)(-\sin t) + (1 + \cos(\sin t)) \cos t \, dt. \]

b) Let \( R \) be the unit disc inside \( C \); 
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{R} \int_{R} (N_x - M_y) \, dA = \int_{R} (0 - 3) \, dA = -3 \text{ area}(R) = -3\pi. \]
\[ (0, 4) \quad c_3 \quad (1, 4) \quad \begin{cases} \mathbf{F} \cdot d\mathbf{r} = \int \int_R (N_x - M_y) \, dx \, dy \\ (0, 0) \quad c_1 \quad (1, 0) \quad = \int \int_R (y + \cos x \cos y - \cos x \cos y) \, dx \, dy = \int \int_R y \, dx \, dy \end{cases} \]

5. a) \[ C_4 \quad c_2 \quad c_1 \quad \int_0^4 \int_0^4 y \, dx \, dy = \int_0^4 y \, dy = \frac{[y^2/2]^4_0}{4} = 8. \]

b) On \( C_4 \), \( \mathbf{F} = \sin y \, \mathbf{i} \), whereas \( d\mathbf{r} \) is parallel to \( \mathbf{j} \). Hence \( \mathbf{F} \cdot d\mathbf{r} = 0 \). Therefore the work of \( \mathbf{F} \) along \( C_4 \) equals 0. Thus

\[
\begin{align*}
\int_{C_1 + C_2 + C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_4} \mathbf{F} \cdot d\mathbf{r} - \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{r}; \\
\text{and the total work along } C_1 + C_2 + C_3 \text{ is equal to the work along } C.
\end{align*}
\]

6. Let \( u = 2x - y \) and \( v = x + y - 1 \). The Jacobian \[
\begin{vmatrix}
\frac{u_x}{u_y} & \frac{u_y}{v_y}
\end{vmatrix} = \left| \begin{array}{cc}
2 & -1 \\
1 & 1
\end{array} \right| = 3.
\]

Hence \( du \, dv = 3 \, dx \, dy \) and \( dx \, dy = \frac{1}{3} \, du \, dv \), so that

\[
V = \int \int_{(2x-y)^2+(x+y-1)^2 < 4} 4 - (2x - y)^2 - (x + y - 1)^2 \, dx \, dy \\
= \int \int_{u^2 + v^2 < 4} (4 - u^2 - v^2) \frac{1}{3} \, du \, dv \\
= \int_0^{2\pi} \int_0^2 (4 - r^2) \frac{1}{3} r \, dr \, d\theta = \int_0^{2\pi} \frac{2}{3} \frac{r^2}{2} - \frac{1}{12} r^3 \bigg|_0^2 \\
= \int_0^{2\pi} \frac{4}{3} \, d\theta = \frac{8\pi}{3}.
\]