I. Limits in Iterated Integrals

For most students, the trickiest part of evaluating multiple integrals by iteration is to put in the limits of integration. Fortunately, a fairly uniform procedure is available which works in any coordinate system. You must always begin by sketching the region; in what follows we'll assume you've done this.

1. Double integrals in rectangular coordinates.

Let's illustrate this procedure on the first case that's usually taken up: double integrals in rectangular coordinates. Suppose we want to evaluate over the region $R$ pictured the integral

$$\int \int_{R} f(x, y) \, dy \, dx$$

where $R = \text{region between } x^2 + y^2 = 1 \text{ and } x + y = 1$; we are integrating first with respect to $y$. Then to put in the limits,

1. Hold $x$ fixed, and let $y$ increase (since we are integrating with respect to $y$). As the point $(x, y)$ moves, it traces out a vertical line.

2. Integrate from the $y$-value where this vertical line enters the region $R$, to the $y$-value where it leaves $R$.

3. Then let $x$ increase, integrating from the lowest $x$-value for which the vertical line intersects $R$, to the highest such $x$-value.

Carrying out this program for the region $R$ pictured, the vertical line enters $R$ where $y = 1 - x$, and leaves where $y = \sqrt{1 - x^2}$.

The vertical lines which intersect $R$ are those between $x = 0$ and $x = 1$. Thus we get for the limits:

$$\int \int_{R} f(x, y) \, dy \, dx = \int_{0}^{1} \int_{1-x}^{\sqrt{1-x^2}} f(x, y) \, dy \, dx.$$

To calculate the double integral, integrating in the reverse order $\int \int_{R} f(x, y) \, dx \, dy$,

1. Hold $y$ fixed, let $x$ increase (since we are integrating first with respect to $x$). This traces out a horizontal line.

2. Integrate from the $x$-value where the horizontal line enters $R$ to the $x$-value where it leaves.

3. Choose the $y$-limits to include all of the horizontal lines which intersect $R$.

Following this prescription with our integral we get:

$$\int \int_{R} f(x, y) \, dx \, dy = \int_{0}^{1} \int_{1-y}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy.$$

Exercises: 3A-2
2. Double integrals in polar coordinates

The same procedure for putting in the limits works for these integrals also. Suppose we want to evaluate over the same region \( R \) as before

\[
\int \int_R \, dr \, d\theta .
\]

As usual, we integrate first with respect to \( r \). Therefore, we

1. Hold \( \theta \) fixed, and let \( r \) increase (since we are integrating with respect to \( r \)). As the point moves, it traces out a ray going out from the origin.

2. Integrate from the \( r \)-value where the ray enters \( R \) to the \( r \)-value where it leaves. This gives the limits on \( r \).

3. Integrate from the lowest value of \( \theta \) for which the corresponding ray intersects \( R \) to the highest value of \( \theta \).

To follow this procedure, we need the equation of the line in polar coordinates. We have

\[
\begin{align*}
x + y &= 1 \\
r \cos \theta + r \sin \theta &= 1, \\
or &= \frac{1}{\cos \theta + \sin \theta}.
\end{align*}
\]

This is the \( r \) value where the ray enters the region; it leaves where \( r = 1 \). The rays which intersect \( R \) lie between \( \theta = 0 \) and \( \theta = \pi/2 \). Thus the double iterated integral in polar coordinates has the limits

\[
\int_0^{\pi/2} \int_{1/(\cos \theta + \sin \theta)}^1 dr \, d\theta .
\]

Exercises: 3B-1

3. Triple integrals in rectangular and cylindrical coordinates.

You do these the same way, basically. To supply limits for \( \iiint_D \, dz \, dy \, dx \) over the region \( D \), we integrate first with respect to \( z \). Therefore we

1. Hold \( x \) and \( y \) fixed, and let \( z \) increase. This gives us a vertical line.

2. Integrate from the \( z \)-value where the vertical line enters the region \( D \) to the \( z \)-value where it leaves \( D \).

3. Supply the remaining limits (in either \( xy \)-coordinates or polar coordinates) so that you include all vertical lines which intersect \( D \). This means that you will be integrating the remaining double integral over the region \( R \) in the \( xy \)-plane which \( D \) projects onto.

For example, if \( D \) is the region lying between the two paraboloids

\[
z = x^2 + y^2 \quad \text{and} \quad z = 4 - x^2 - y^2,
\]

we get by following steps 1 and 2,

\[
\int \int_D \, dz \, dy \, dx = \int \int_R \int_{z = x^2 + y^2}^{z = 4 - x^2 - y^2} \, dz \, dA .
\]
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where $R$ is the projection of $D$ onto the $xy$-plane. To finish the job, we have to determine what this projection is. From the picture, what we should determine is the $xy$-curve over which the two surfaces intersect. We find this curve by eliminating $z$ from the two equations, getting

\[ x^2 + y^2 = 4 - x^2 - y^2, \quad \text{which implies} \]
\[ x^2 + y^2 = 2. \]

Thus the $xy$-curve bounding $R$ is the circle in the $xy$-plane with center at the origin and radius $\sqrt{2}$.

This makes it natural to finish the integral in polar coordinates. We get

\[ \int \int_D dz \, dy \, dx = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{z^2+y^2}^{4-z^2-y^2} dz \, r \, dr \, d\theta; \]

the limits on $z$ will be replaced by $r^2$ and $4 - r^2$ when the integration is carried out.

Exercises: 5A-2

4. Spherical coordinates.

Once again, we use the same procedure. To calculate the limits for an iterated integral $\iiint_D d\rho \, d\phi \, d\theta$ over a region $D$ in 3-space, we are integrating first with respect to $\rho$. Therefore we

1. Hold $\phi$ and $\theta$ fixed, and let $\rho$ increase. This gives us a ray going out from the origin.

2. Integrate from the $\rho$-value where the ray enters $D$ to the $\rho$-value where the ray leaves $D$. This gives the limits on $\rho$.

3. Hold $\theta$ fixed and let $\phi$ increase. This gives a family of rays, that form a sort of fan. Integrate over those $\phi$-values for which the rays intersect the region $D$.

4. Finally, supply limits on $\theta$ so as to include all of the fans which intersect the region $D$.

For example, suppose we start with the circle in the $yz$-plane of radius 1 and center at $(1,0)$, rotate it about the $z$-axis, and take $D$ to be that part of the resulting solid lying in the first octant.

First of all, we have to determine the equation of the surface formed by the rotated circle. In the $yz$-plane, the two coordinates $\rho$ and $\phi$ are indicated. To see the relation between them when $P$ is on the circle, we see that also angle $OAP = \phi$, since both the angle $\phi$ and $OAP$ are complements of the same angle, $AO$. From the right triangle, this shows the relation is $\rho = 2 \sin \phi$.

As the circle is rotated around the $z$-axis, the relationship stays the same, so $\rho = 2 \sin \phi$ is the equation of the whole surface.

To determine the limits of integration, when $\phi$ and $\theta$ are fixed, the corresponding ray enters the region where $\rho = 0$ and leaves where $\rho = 2 \sin \phi$.

As $\phi$ increases, with $\theta$ fixed, it is the rays between $\phi = 0$ and $\phi = \pi/2$ that intersect $D$, since we are only considering the portion of the surface lying in the first octant (and thus above the $xy$-plane).
Again, since we only want the part in the first octant, we only use $\theta$ values from 0 to $\pi/2$. So the iterated integral is

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2 \sin \phi} d\rho \, d\phi \, d\theta.$$ 

Exercises: 5B-1