FIRST MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature:

Recitation Time:

There are 5 problems, and the total number of points is 100. Show all your work. Please make your work as clear and easy to follow as possible.

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1. (20pts) (i) Suppose that the four vectors \( \vec{t}, \vec{u}, \vec{v} \) and \( \vec{w} \) lie in the same plane \( \Pi \). Show that
\[
(\vec{t} \times \vec{u}) \times (\vec{v} \times \vec{w}) = \vec{0}.
\]
\( \vec{E} \times \vec{W} \) is orthogonal to both \( \vec{t} \) and \( \vec{u} \).
So \( \vec{E} \times \vec{W} \) is normal to the plane \( \Pi \).
Similarly \( \vec{V} \times \vec{W} \) is orthogonal to both \( \vec{V} \) and \( \vec{W} \).
So \( \vec{V} \times \vec{W} \) is normal to the plane \( \Pi \).
But then \( \vec{E} \times \vec{W} \) and \( \vec{V} \times \vec{W} \) are parallel so that
\[
(\vec{E} \times \vec{W}) \times (\vec{V} \times \vec{W}) = \vec{0}.
\]

(ii) Now suppose that \( \vec{t}, \vec{u}, \vec{v} \) and \( \vec{w} \) are four non-zero vectors in \( \mathbb{R}^3 \), such that
\[
(\vec{t} \times \vec{u}) \times (\vec{v} \times \vec{w}) = \vec{0}.
\]
Is it true that these four vectors have to lie in the same plane? If true, explain why and if false, give a counterexample.

No, it is not true.
Take \( \vec{E} = \vec{u} = \hat{i}, \vec{V} = \hat{j}, \vec{W} = \hat{k} \).
Then these vectors don't lie in the same plane, so
\[
(\vec{E} \times \vec{W}) \times (\vec{V} \times \vec{W}) = \hat{i} \times \hat{i} = \vec{0},
\]
so that
\[
(\vec{E} \times \vec{W}) \times (\vec{V} \times \vec{W}) = \vec{0}.
\]
2. (20pts) (i) Find a parametric equation for the line \( l \) through the two points \( P = (1, -1, 2) \) and \( Q = (-1, 3, 3) \).

\[
\overrightarrow{PQ} = (-2, 4, 1) \quad \text{If} \quad R = (x, y, z) \quad \text{is any pt. on the line, then} \quad \overrightarrow{PR} = t \overrightarrow{PQ}, \text{some} \ t.
\]

So
\[
(x-1, y+1, z-2) = t(-2, 4, 1)
\]

\[
(x, y, z) = (1 - 2t, 4t - 1, t + 2)
\]

(ii) Find the distance between the line \( l \) and the line \( m \) given parametrically by \((x, y, z) = (t - 1, 2t + 1, 3 - t)\).

\[
\overrightarrow{v} = (-1, 4, 1) \quad \overrightarrow{w} = (1, 2, -1)
\]

normal \( \overrightarrow{n} \) to both lines \( = \overrightarrow{v} \times \overrightarrow{w} = \begin{vmatrix} 
\hat{i} & \hat{j} & \hat{k} \\
-1 & 4 & 1 \\
1 & 2 & -1 \\
\end{vmatrix} = 6 \hat{i} - 6 \hat{j} + 8 \hat{k}
\]

Take \( \overrightarrow{n}' = 6 \hat{i} + \hat{j} + 8 \hat{k} \)

\( P = (1, -1, 2) \quad P' = (-1, 1, 3) \quad \text{pts on both lines} \)

\( \overrightarrow{RR'} \) closest pt. then \( \overrightarrow{RR'} = \text{proj}_{\overrightarrow{n}'} \overrightarrow{PP'} = \frac{-2}{101} (6, 1, 8) \)

\( \overrightarrow{PP'} = (-2, 2, 1) \)

\[
\overrightarrow{n}' \cdot \overrightarrow{PP'} = 6t + 1 + 8
\]

\[
||\overrightarrow{n}'||^2 = 36 + 1 + 64 = 101
\]

\[
d = \frac{2}{101} \times (101) = \frac{2}{101} \times 101 = 2
\]
3. (20pts) (i) Find the volume of the parallelepiped spanned by the vectors \( \vec{u} = (1, 2, -3) \), \( \vec{v} = (1, -2, 1) \) and \( \vec{w} = (-1, -2, -1) \).

Signed volume is equal to the scalar triple product \( (\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ -1 & 2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 0 + 12 = 16 \)

Volume = 16

(ii) Do the vectors \( \vec{u} \), \( \vec{v} \) and \( \vec{w} \) form a right-handed set or a left-handed set?

Sign of scalar triple product is the, so we have a right-handed set.
4. (20pts) Let $D$ be the region inside the sphere of radius $2a$ centred at the origin and outside the cylinder of radius $a$ centred around the $z$-axis.

(i) Describe the region $D$ in cylindrical coordinates.

\[
\text{outside the cylinder of radius } a : \quad r > a, \quad r^2 + z^2 \leq a^2
\]
\[
\text{inside the sphere of radius } 2a : \quad x^2 + y^2 + z^2 \leq a^2
\]
\[
\quad r \geq a, \quad r^2 + z^2 \leq a^2
\]

(ii) Describe the region $D$ in spherical coordinates.

\[
\text{inside the sphere of radius } 2a : \quad \rho \leq 2a
\]
\[
\text{outside the cylinder of radius } a : \quad z \cos \phi > a
\]
\[
\quad \rho \leq 2a, \quad z \cos \phi > a.
\]
5. (20pts) Determine whether or not the following limits exist, and if they do exist, then find the limit. Explain your answer.

(i) \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \).

No, limit does not exist.

If we approach \((0,0)\) along line \(x = 0\) we get \( \lim_{y \to 0} \frac{y}{y^2} = 0 \).

If we approach \((0,0)\) along line \(y = x\) we get \( \lim_{x \to 0} \frac{x^2}{2x} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2} \neq 0 \). So limit does not exist.

(ii) \( \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \).

Yes, limit does exist. Use polar coordinates \( x = \rho \cos \theta, y = \rho \sin \theta \) \( \sqrt{x^2 + y^2} = \rho \).

So \( \lim_{(x,y) \to (0,0)} \frac{\rho^2 \cos \theta \sin \theta}{\rho} = \lim_{\rho \to 0} \left| \rho \cos \theta \sin \theta \right| \leq \lim_{\rho \to 0} |\rho| = 0. \)