THIRD MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature:

Recitation Time:

There are 5 problems, and the total number of points is 100. Show all your work. Please make your work as clear and easy to follow as possible.

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1. (20 pts) For what values of $\lambda$ does the function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$,
\[ f(x, y, z) = \lambda x^2 - \lambda xy + y^2 + \lambda z^2, \]
have a non-degenerate local minimum at $(0, 0, 0)$?

\[ \nabla f = (2\lambda x - \lambda y, 2y - \lambda x, 2\lambda z) \]

\[ H_f = \begin{pmatrix} 2\lambda & -\lambda & 0 \\ -\lambda & 2 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} \]

\[ d_1 = 2\lambda, \quad d_2 = 4\lambda - \lambda^2, \quad d_3 = 2\lambda d_2 \]

Minimum: $d_1 > 0$, $d_2 > 0$, $d_3 > 0$

So, $\lambda > 0$, $4\lambda - \lambda^2 > 0$, $\lambda > 0$.

$\lambda(4-\lambda) > 0$

$\lambda < 4$

$0 < \lambda < 4$. $\lambda \in (0, 4)$. 
2. (20pts) Let \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) be the function \( f(x, y, z) = x^2 - y^2 + z^2 \).

(i) Show that \( f \) has a global maximum on the ellipsoid \( 2x^2 + 3y^2 + z^2 = 6 \).

\[ K = \{ (x, y, z) \in \mathbb{R}^3 \mid 2x^2 + 3y^2 + z^2 = 6 \} \] is closed and bounded. So \( K \) is compact.

\( f \) is \( \text{c} \), \( K \) is compact \( \Rightarrow \) \( f \) has a global maximum.

(ii) Find this maximum.

Consider \( h: \mathbb{R}^4 \rightarrow \mathbb{R} \) given by \( h(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(2x^2 + 3y^2 + z^2 - 6) \).

Critical pts of \( h \): \( 2x = -4\lambda x \)
\( 2y = 4\lambda y \)
\( 2z = 2\lambda z \)
\( 2x^2 + 3y^2 + z^2 = 6 \).

Either \( x = y = 0, \lambda = 1; y = z = 0, \lambda = -\frac{1}{2}; x = z = 0, \lambda = \lambda \frac{1}{2} \)

\( x = y = 0, z = \sqrt{6}; y = z = 0, x = \sqrt{3}; x = z, y = \sqrt{2} \)

Of these three pts, \( (0, 0, \sqrt{6}) \) gives biggest pt.

Absolute maximum is \( 6 \).
3. (20pts)

(i) Switch the order of integration in the integral
\[ \int_0^3 \int_x^9 xe^{-y^2} \, dy \, dx. \]

(ii) Evaluate this integral.
\[ \int_0^9 e^{-y^2} \left( \int_0^{\sqrt{x^2+y^2}} x \, dx \right) \, dy = \int_0^9 e^{-y^2} \left[ \frac{x^2}{2} \right]_0^{\sqrt{x^2+y^2}} \, dy \]
\[ = -\frac{1}{4} \int_0^9 2ye^{-y^2} \, dy \]
\[ = -\frac{1}{4} \left[ e^{-y^2} \right]_0^9 \]
\[ = \frac{1}{4} \left( 1 - e^{-81} \right). \]
4. (20pts) Let \( W \) be the region inside the sphere \( x^2 + y^2 + z^2 = 1 \) and inside the cone \( z^2 = x^2 + y^2 \).
Set up an integral to calculate the integral of the function \( yz \) over \( W \) and calculate this integral.

View \( W \) as a region of type I.

\[
\iiint_{W} yz \, dz \, dy \, dx = \int_{\frac{1}{\sqrt{2}}}^{1} \left( \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \left( \int_{\frac{x+y^2}{2}}^{\frac{1-x^2+y^2-x^2-y^2}{2}} dy \right) dy \right) dx
\]

\[
= \int_{\frac{1}{\sqrt{2}}}^{1} \left( \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \left( \frac{1-x^2-y^2-x^2-y^2}{2} \right) dy \right) dx
\]

\[
= \int_{\frac{1}{\sqrt{2}}}^{1} 0 \, dx = 0
\]

As \( y, y^3 \) are odd functions.

In retrospect, \( J(y) = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{z}{\sqrt{x^2+y^2}} \, dz \) is an even function \( y \), so that \( yJ(y) \) is an odd function. Therefore integral is zero.
5. (20 pts) Let \( D \) be the region in the first quadrant bounded by the curves \( x^2 - y^2 = 1, x^2 - y^2 = 4, xy = 1 \) and \( xy = 3 \).

(i) Find \( du \, dv \) in terms of \( dx \, dy \), where \( u = x^2 - y^2 \) and \( v = xy \).

\[
\frac{\partial (uv)}{\partial (xy)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2) \quad \frac{\partial (x^2 + y^2)}{\partial (uv)} = \frac{1}{2(x^2 + y^2)} \\

du \, dv = 2(x^2 + y^2) \, dx \, dy
\]

(ii) Evaluate the integral

\[
\iint_D (x^4 - y^4) \, dx \, dy.
\]

\[
\int_1^3 \int_1^4 \frac{u}{2} \, du \, dv = \frac{1}{4} \int_1^3 \left[ u^2 \right]^4_1 \, dv
\]

\[
= \frac{1}{4} \int_1^3 15 \, dv
\]

\[
= \frac{15}{2}
\]