16. Moving frames

Definition 16.1. We say a parametrised differentiable curve \( \vec{r} : I \rightarrow \mathbb{R}^n \) is **regular** if \( \vec{r}'(t) \neq 0 \) (the speed is never zero).

We say that \( \vec{r}(t) \) is **smooth** if \( \vec{r}(t) \) is \( \mathcal{C}^\infty \).

Given a regular smooth parametrised differentiable curve \( \vec{r} : I \rightarrow \mathbb{R}^3 \), we can parametrise by arclength, in which case we get can write down the unit tangent vector

\[
\vec{T} = \frac{d\vec{r}}{ds}(s).
\]

The curvature \( \kappa(s) \) is defined as the magnitude of

\[
\frac{d\vec{T}}{ds}(s).
\]

If the curvature is nowhere zero, then we define the normal vector \( \vec{N}(s) \) as the unit vector pointing in the direction of the derivative of the tangent vector:

\[
\frac{d\vec{T}}{ds}(s) = \kappa(s)\vec{N}(s).
\]

We have already seen that \( \vec{T}(s) \) and \( \vec{N}(s) \) are orthogonal.

Definition 16.2. \( \vec{B}(s) = \vec{T}(s) \times \vec{N}(s) \)

is called the **binormal vector**.

The three vectors \( \vec{T}(s) \), \( \vec{N}(s) \), and \( \vec{B}(s) \) are unit vectors and pairwise orthogonal, that is, these vectors are an orthonormal basis of \( \mathbb{R}^3 \). Notice that \( \vec{T}(s) \), \( \vec{N}(s) \), and \( \vec{B}(s) \) are a right handed set.

We call these vectors a **moving frame** or the **Frenet-Serret frame**.

Now

\[
\frac{d\vec{B}}{ds}(s) \times \vec{B}(s) = 0,
\]

as

\[
||\vec{B}(s) \cdot \vec{B}(s)|| = 1.
\]

It follows that

\[
\frac{d\vec{B}}{ds}(s),
\]
lies in the plane spanned by $\vec{T}(s)$ and $\vec{N}(s)$.

\[
\frac{d\vec{B}}{ds}(s) \cdot \vec{T}(s) = \frac{d(\vec{T} \times \vec{N})}{ds}(s) \cdot \vec{T}(s)
= \left( \frac{d\vec{T}}{ds}(s) \times \vec{N}(s) + \vec{T}(s) \times \frac{d\vec{N}}{ds}(s) \right) \cdot \vec{T}(s)
= \kappa(s)(\vec{N}(s) \times \vec{N}(s)) \cdot \vec{T}(s) + (\vec{T}(s) \times \vec{N}(s)) \cdot \vec{T}(s)
= 0 + (\vec{T}(s) \times \vec{T}(s)) \cdot \vec{N}(s)
= 0.
\]

It follows that

\[
\frac{d\vec{B}}{ds}(s) \quad \text{and} \quad \vec{T}(s),
\]

are orthogonal, and so

\[
\frac{d\vec{B}}{ds}(s) \quad \text{is parallel to} \quad \vec{N}(s).
\]

**Definition 16.3.** The **torsion** of the curve $\vec{r}(s)$ is the unique scalar $\tau(s)$ such that

\[
\frac{d\vec{B}}{ds}(s) = -\tau(s)\vec{N}(s).
\]

If we have a helix, the sign of the torsion distinguishes between a right handed helix and a left handed helix. The magnitude of the torsion measures how spread out the helix is (the curvature measures how tight the turns are). Now

\[
\frac{d\vec{N}}{ds}(s)
\]

is orthogonal to $\vec{N}(s)$, and so it is a linear combination of $\vec{T}(s)$ and $\vec{B}(s)$. In fact,

\[
\frac{d\vec{N}}{ds}(s) = \frac{d(\vec{B} \times \vec{T})}{ds}(s)
= \frac{d\vec{B}}{ds}(s) \times \vec{T}(s) + \vec{B}(s) \times \frac{d\vec{T}}{ds}(s)
= -\tau(s)\vec{N}(s) \times \vec{T}(s) + \kappa(s)\vec{B}(s) \times \vec{N}(s)
= \tau(s)\vec{B}(s) - \kappa(s)\vec{T}(s)
= -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s).
\]
Theorem 16.4 (Frenet Formulae). Let $\vec{r}: I \rightarrow \mathbb{R}^3$ be a regular smooth parametrised curve. Then

$$
\begin{pmatrix}
\vec{T}'(s) \\
\vec{N}'(s) \\
\vec{B}'(s)
\end{pmatrix} =
\begin{pmatrix}
0 & \kappa(s) & 0 \\
-\kappa(s) & 0 & \tau(s) \\
0 & -\tau(s) & 0
\end{pmatrix}
\begin{pmatrix}
\vec{T}(s) \\
\vec{N}(s) \\
\vec{B}(s)
\end{pmatrix}.
$$

Of course, $s$ represents the arclength parameter and primes denote derivatives with respect to $s$. Notice that the $3 \times 3$ matrix $A$ appearing in (16.4) is skew-symmetric, that is $A^t = -A$. The way we have written the Frenet formulae, it appears that we have two $3 \times 1$ vectors; strictly speaking these are the rows of two $3 \times 3$ matrices.

Theorem 16.5. Let $I \subset \mathbb{R}$ be an open interval and suppose we are given two smooth functions $\kappa: I \rightarrow \mathbb{R}$ and $\tau: I \rightarrow \mathbb{R}$, where $\kappa(s) > 0$ for all $s \in I$.

Then there is a regular smooth curve $\vec{r}: I \rightarrow \mathbb{R}^3$ parametrised by arclength with curvature $\kappa(s)$ and torsion $\tau(s)$. Further, any two such curves are congruent, that is, they are the same up to translation and rotation.

Remark 16.6. Uniqueness is one of the hwk problems.

Let's consider the example of the helix:

Example 16.7.

$$
\vec{r}(s) = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, b s),
$$

where

$$
c^2 = a^2 + b^2.
$$

Let's assume that $a > 0$. By convention $c > 0$. Then

$$
\vec{T}(s) = \frac{1}{c}(-a \sin \frac{s}{c}, a \cos \frac{s}{c}, b).
$$

Hence

$$
\frac{dT}{ds}(s) = \frac{-a}{c^2}(\cos \frac{s}{c}, \sin \frac{s}{c}, 0) = \frac{a}{c^2}(-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0) = \frac{a}{c^2} \vec{N}(s)
$$

It follows that

$$
\kappa(s) = \frac{a}{c^2} \quad \text{and} \quad \vec{N}(s) = (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0).
$$

Finally,

$$
\vec{B}(s) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-\frac{a}{c} \sin \frac{s}{c} & \frac{a}{c} \cos \frac{s}{c} & b \\
-\cos \frac{s}{c} & -\sin \frac{s}{c} & 0
\end{vmatrix}
$$
It follows that
\[ \tilde{B}(s) = \left( \frac{b}{c} \sin \frac{s}{c}, \frac{-b}{c} \cos \frac{s}{c}, a \right) = \frac{1}{c} \left( b \sin \frac{s}{c}, -b \cos \frac{s}{c}, a \right). \]

Finally, note that
\[ \frac{d\tilde{B}}{ds}(s) = \frac{b}{c^2} \left( \cos \frac{s}{c}, \sin \frac{s}{c}, 0 \right) = -\frac{b}{c^2} \tilde{N}. \]

Using this we can compute the torsion:
\[ \tau(s) = \frac{b}{c^2}. \]

It is interesting to use the torsion and curvature to characterise various geometric properties of curves. Let’s say that a parametrised differentiable curve \( \tilde{r} : I \rightarrow \mathbb{R}^3 \) is **planar** if there is a plane \( \Pi \) which contains the image of \( \tilde{r} \).

**Theorem 16.8.** A regular smooth curve \( \tilde{r} : I \rightarrow \mathbb{R}^3 \) is planar if and only if the torsion is zero.

**Proof.** We may assume that the curve passes through the origin.

Suppose that \( \tilde{r} \) is planar. Then the image of \( \tilde{r} \) is contained in a plane \( \Pi \). As the curve passes through the origin, \( \Pi \) contains the origin as well. Note that the unit tangent vector \( \tilde{T}(s) \) and the unit normal vector \( \tilde{N}(s) \) are contained in \( \Pi \). It follows that \( \tilde{B}(s) \) is a normal vector to the plane; as \( \tilde{B}(s) \) is a unit vector, it must be constant. But then
\[ \frac{d\tilde{B}}{ds}(s) = \tilde{0} = 0\tilde{N}(s), \]
so that the torsion is zero.

Now suppose that the torsion is zero. Then
\[ \frac{d\tilde{B}}{ds}(s) = 0\tilde{N} = \tilde{0}, \]
so that \( \tilde{B}(s) = B_0 \), is a constant vector. Consider the function
\[ f(s) = \tilde{r}(s) \cdot \tilde{B}(s) = \tilde{r}(s) \cdot \tilde{B}_0. \]

Then
\[ \frac{df}{ds}(s) = \frac{d(\tilde{r} \times \tilde{B}_0)}{ds}(s) = \tilde{T}(s) \cdot \tilde{B}_0 = 0. \]

So \( f(s) \) is constant. It is zero when \( \tilde{r}(a) = \tilde{0} \) (the curve passes through the origin) so that \( f(s) = 0 \). But then \( \tilde{r}(s) \) is always orthogonal to a fixed vector, so that \( \tilde{r} \) is contained in a plane, that is, \( C \) is planar. \( \square \)
It is interesting to try to figure out how to characterise curves which are contained in spheres or cylinders.
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