17. Vector fields

Definition 17.1. Let $A \subset \mathbb{R}^n$ be an open subset. A vector field on $A$ is function $\vec{F}: A \rightarrow \mathbb{R}^n$.

One obvious way to get a vector field is to take the gradient of a differentiable function. If $f: A \rightarrow \mathbb{R}$, then

$$\nabla f: A \rightarrow \mathbb{R}^n,$$

is a vector field.

Definition 17.2. A vector field $\vec{F}: A \rightarrow \mathbb{R}^n$ is called a gradient (aka conservative) vector field if $\vec{F} = \nabla f$ for some differentiable function $f: A \rightarrow \mathbb{R}$.

Example 17.3. Let

$$\vec{F}: \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}^3,$$

be the vector field

$$\vec{F}(x, y, z) = \frac{cx}{(x^2 + y^2 + z^2)^{3/2}} \hat{i} + \frac{cy}{(x^2 + y^2 + z^2)^{3/2}} \hat{j} + \frac{cz}{(x^2 + y^2 + z^2)^{3/2}} \hat{k},$$

for some constant $c$. Then $\vec{F}(x, y, z)$ is the gradient of

$$f: \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R},$$

given by

$$f(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{1/2}}.$$

So $\vec{F}$ is a conservative vector field. Notice that if $c < 0$ then $\vec{F}$ models the gravitational force and $f$ is the potential (note that unfortunately mathematicians and physicists have different sign conventions for $f$).

Proposition 17.4. If $\vec{F}$ is a conservative vector field and $\vec{F}$ is $C^1$ function, then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

for all $i$ and $j$ between 1 and $n$.

Proof. If $\vec{F}$ is conservative, then we may find a differentiable function $f: A \rightarrow \mathbb{R}^n$ such that

$$F_i = \frac{\partial f}{\partial x_i}.$$
As \( F_i \) is \( C^1 \) for each \( i \), it follows that \( f \) is \( C^2 \). But then
\[
\frac{\partial F_i}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial F_j}{\partial x_i}.
\]

Notice that (17.4) is a negative result; one can use it show that various vector fields are not conservative.

**Example 17.5.** Let
\[
\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (-y, x).
\]
Then
\[
\frac{\partial F_1}{\partial y} = -1 \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1 \neq -1.
\]
So \( \vec{F} \) is not conservative.

**Example 17.6.** Let
\[
\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (y, x + y).
\]
Then
\[
\frac{\partial F_1}{\partial y} = 1 \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1,
\]
so \( \vec{F} \) might be conservative. Let’s try to find
\[
f : \mathbb{R}^2 \rightarrow \mathbb{R} \quad \text{such that} \quad \nabla f(x, y) = (y, x + y).
\]
If \( f \) exists, then we must have
\[
\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x + y.
\]
If we integrate the first equation with respect to \( x \), then we get
\[
f(x, y) = xy + g(y).
\]
Note that \( g(y) \) is not just a constant but it is a function of \( y \). There are two ways to see this. One way, is to imagine that for every value of \( y \), we have a separate differential equation. If we integrate both sides, we get an arbitrary constant \( c \). As we vary \( y \), \( c \) varies, so that \( c = g(y) \) is a function of \( y \). On the other hand, if to take the partial derivatives
of $g(y)$ with respect to $x$, then we get $0$. Now we take $xy + g(y)$ and differentiate with respect to $y$, to get

$$x + y = \frac{\partial(xy + g(y))}{\partial y} = x + \frac{dg}{dy}(y).$$

So

$$g'(y) = y.$$

Integrating both sides with respect to $y$ we get

$$g(y) = \frac{y^2}{2} + c.$$

It follows that

$$\nabla(xy + \frac{y^2}{2}) = (y, x + y),$$

so that $\vec{F}$ is conservative.

**Definition 17.7.** If $\vec{F}: A \rightarrow \mathbb{R}^n$ is a vector field, we say that a parametrised differentiable curve $\vec{r}: I \rightarrow A$ is a flow line for $\vec{F}$, if

$$\vec{r}'(t) = \vec{F}(\vec{r}(t)),$$

for all $t \in I$.

**Example 17.8.** Let

$$\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (-y, x).$$

We check that

$$\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{r}(t) = (a \cos t, a \sin t),$$

is a flow line. In fact

$$\vec{r}'(t) = (-a \sin t, a \cos t),$$

and so

$$\vec{F}(\vec{r}(t)) = \vec{F}(a \cos t, a \sin t)$$

$$= \vec{r}'(t),$$

so that $\vec{r}(t)$ is indeed a flow line.

**Example 17.9.** Let

$$\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (-x, y).$$

Let’s find a flow line through the point $(a, b)$. We have

$$x'(t) = -x(t) \quad \text{with} \quad x(0) = a$$

$$y'(t) = y(t) \quad \text{with} \quad y(0) = b.$$

Therefore,

$$x(t) = ae^{-t} \quad \text{and} \quad y(t) = be^t,$$

gives the flow line through $(a, b)$.  

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Example 17.10. Let
\[ \vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \]
given by \( \vec{F}(x, y) = (x^2 - y^2, 2xy) \).

Try
\[
\begin{align*}
x(t) &= 2a \cos t \sin t \\
y(t) &= 2a \sin^2 t.
\end{align*}
\]

Then
\[
\begin{align*}
x'(t) &= 2a(- \sin^2 t + \cos^t) \\
&= \frac{x^2(t) - y^2(t)}{y(t)}.
\end{align*}
\]

Similarly
\[
\begin{align*}
y'(t) &= 4a \cos t \sin t \\
&= \frac{2x(t)y(t)}{y(t)}.
\end{align*}
\]

So
\[
\vec{r}'(t) = \frac{\vec{F}(\vec{r}(t))}{f(t)}.
\]

So the curves themselves are flow lines, but this is not the correct parametrisation. The flow lines are circles passing through the origin, with centre along the y-axis.

Example 17.11. Let
\[ \vec{F} : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}^2 \]
given by \( \vec{F}(x, y) = (-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}) \).

Then
\[
\frac{\partial F_1}{\partial y}(x, y) = -\frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},
\]
and
\[
\frac{\partial F_2}{\partial x}(x, y) = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.
\]

So \( \vec{F} \) might be conservative. Let’s find the flow lines. Try
\[
\begin{align*}
x(t) &= a \cos \left( \frac{t}{a^2} \right) \\
y(t) &= a \sin \left( \frac{t}{a^2} \right).
\end{align*}
\]
Then

\[ x'(t) = -\frac{1}{a} \sin \left( \frac{t}{a^2} \right) \]
\[ = -\frac{y}{x^2 + y^2}. \]

Similarly

\[ y'(t) = \frac{1}{a} \cos \left( \frac{t}{a^2} \right) \]
\[ = \frac{x}{x^2 + y^2}. \]

So the flow lines are closed curves. In fact this means that \( \vec{F} \) is not conservative.