24. Triple integrals

Definition 24.1. Let \( B = [a, b] \times [c, d] \times [e, f] \subset \mathbb{R}^3 \) be a box in space. A partition \( \mathcal{P} \) of \( \mathbb{R} \) is a triple of sequences:
\[
a = x_0 < x_1 < \cdots < x_n = b \\
c = y_0 < y_1 < \cdots < y_n = d \\
e = z_0 < z_1 < \cdots < z_n = f.
\]

The mesh of \( \mathcal{P} \) is
\[m(\mathcal{P}) = \max\{x_i - x_{i-1}, y_i - y_{i-1}, z_i - z_{i-1} | 1 \leq i \leq k \}\].

Now suppose we are given a function
\[f : B \rightarrow \mathbb{R}\]

Pick
\[
\bar{c}_{ijk} \in B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{i-1}, z_i].
\]

Definition 24.2. The sum
\[S = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(\bar{c}_{ijk})(x_i - x_{i-1})(y_j - y_{j-1})(z_i - z_{i-1}),\]
is called a Riemann sum.

Definition 24.3. The function \( f : B \rightarrow \mathbb{R} \) is called integrable, with integral \( I \), if for every \( \epsilon > 0 \), we may find a \( \delta > 0 \) such that for every mesh \( \mathcal{P} \) whose mesh size is less than \( \delta \), we have
\[|I - S| < \epsilon,\]
where \( S \) is any Riemann sum associated to \( \mathcal{P} \).

If \( W \subset \mathbb{R}^3 \) is a bounded subset and \( f : W \rightarrow \mathbb{R} \) is a bounded function, then pick a box \( B \) containing \( W \) and extend \( f \) by zero to a function \( \tilde{f} : B \rightarrow \mathbb{R}, \)
\[
\tilde{f}(x) = \begin{cases} 
  x & \text{if } x \in W \\
  0 & \text{otherwise.}
\end{cases}
\]

If \( \tilde{f} \) is integrable, then we write
\[
\iiint_{W} f(x, y, z) \, dx \, dy \, dz = \iiint_{B} \tilde{f}(x, y, z) \, dx \, dy \, dz.
\]
In particular
\[\text{vol}(W) = \iiint_{W} \, dx \, dy \, dz.\]

There are two pairs of results, which are much the same as the results for double integrals:
Proposition 24.4. Let $W \subset \mathbb{R}^2$ be a bounded subset and let $f : W \rightarrow \mathbb{R}$ and $g : W \rightarrow \mathbb{R}$ be two integrable functions. Let $\lambda$ be a scalar.

Then
(1) $f + g$ is integrable over $W$ and
$$\iiint_W f(x, y, z) + g(x, y, z) \, dx \, dy \, dz = \iiint_W f(x, y, z) \, dx \, dy \, dz + \iiint_W g(x, y, z) \, dx \, dy \, dz.$$ 
(2) $\lambda f$ is integrable over $W$ and
$$\iiint_W \lambda f(x, y, z) \, dx \, dy \, dz = \lambda \iiint_W f(x, y, z) \, dx \, dy \, dz.$$ 
(3) If $f(x, y, z) \leq g(x, y, z)$ for any $(x, y, z) \in W$, then
$$\iiint_W f(x, y, z) \, dx \, dy \, dz \leq \iiint_W g(x, y, z) \, dx \, dy \, dz.$$ 
(4) $|f|$ is integrable over $W$ and
$$|\iiint_W f(x, y, z) \, dx \, dy \, dz| \leq \iiint_W |f(x, y, z)| \, dx \, dy \, dz.$$ 

Proposition 24.5. Let $W = W_1 \cup W_2 \subset \mathbb{R}^3$ be a bounded subset and let $f : W \rightarrow \mathbb{R}$ be a bounded function.

If $f$ is integrable over $W_1$ and over $W_2$, then $f$ is integrable over $W$ and $W_1 \cap W_2$, and we have
$$\iiint_W f(x, y, z) \, dx \, dy \, dz = \iiint_{W_1} f(x, y, z) \, dx \, dy \, dz + \iiint_{W_2} f(x, y, z) \, dx \, dy \, dz$$
$$- \iiint_{W_1 \cap W_2} f(x, y, z) \, dx \, dy \, dz.$$ 

Definition 24.6. Define three maps
$$\pi_{ij} : \mathbb{R}^3 \rightarrow \mathbb{R}^2,$$
by projection onto the $i$th and $j$th coordinate.

In coordinates, we have
$$\pi_{12}(x, y, z) = (x, y), \quad \pi_{23}(x, y, z) = (y, z), \quad \text{and} \quad \pi_{13}(x, y, z) = (x, z).$$

For example, if we start with a solid pyramid and project onto the $xy$-plane, the image is a square, but it project onto the $xz$-plane, the image is a triangle. Similarly onto the $yz$-plane.

Definition 24.7. A bounded subset $W \subset \mathbb{R}^3$ is an elementary subset if it is one of four types:

**Type 1:** $D = \pi_{12}(W)$ is an elementary region and
$$W = \{ (x, y, z) \in \mathbb{R}^2 \mid (x, y) \in D, c(x, y) \leq z \leq \phi(x, y) \},$$
where \( \epsilon : D \rightarrow \mathbb{R} \) and \( \phi : D \rightarrow \mathbb{R} \) are continuous functions.

**Type 2:** \( D = \pi_{23}(W) \) is an elementary region and
\[
W = \{(x, y, z) \in \mathbb{R}^2 | (y, z) \in D, \alpha(y, z) \leq x \leq \beta(y, z) \},
\]
where \( \alpha : D \rightarrow \mathbb{R} \) and \( \beta : D \rightarrow \mathbb{R} \) are continuous functions.

**Type 3:** \( D = \pi_{13}(W) \) is an elementary region and
\[
W = \{(x, y, z) \in \mathbb{R}^2 | (x, z) \in D, \gamma(x, z) \leq y \leq \delta(x, z) \},
\]
where \( \gamma : D \rightarrow \mathbb{R} \) and \( \delta : D \rightarrow \mathbb{R} \) are continuous functions.

**Type 4:** \( W \) is of type 1, 2 and 3.

The solid pyramid is of type 4.

**Theorem 24.8.** Let \( W \subset \mathbb{R}^3 \) be an elementary region and let \( f : W \rightarrow \mathbb{R} \) be a continuous function.

Then

1. If \( W \) is of type 1, then
\[
\iiint_W f(x, y, z) \, dx \, dy \, dz = \iiint_{\pi_{12}(W)} \left( \int_{\epsilon(x,y)}^{\phi(x,y)} f(x, y, z) \, dz \right) \, dx \, dy.
\]

2. If \( W \) is of type 2, then
\[
\iiint_W f(x, y, z) \, dx \, dy \, dz = \iiint_{\pi_{23}(W)} \left( \int_{\alpha(y,z)}^{\beta(y,z)} f(x, y, z) \, dx \right) \, dy \, dz.
\]

3. If \( W \) is of type 3, then
\[
\iiint_W f(x, y, z) \, dx \, dy \, dz = \iiint_{\pi_{13}(W)} \left( \int_{\gamma(x,z)}^{\delta(x,z)} f(x, y, z) \, dy \right) \, dx \, dz.
\]

Let’s figure out the volume of the solid ellipsoid:
\[
W = \{(x, y, z) \in \mathbb{R}^3 | \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 + \left( \frac{z}{c} \right)^2 \leq 1 \}.
\]
This is an elementary region of type 4.

\[
\text{vol}(W) = \iiint_{W} dx \, dy \, dz
\]

\[
= \int_{-a}^{a} \left( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} \left( \int_{-c\sqrt{1-(\frac{x}{a})^2}}^{c\sqrt{1-(\frac{x}{a})^2}} dy \right) \, dz \right) \, dx
\]

\[
= \int_{-a}^{a} \left( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} 2c \sqrt{1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2} \, dy \right) \, dx
\]

\[
= 2c \int_{-a}^{a} \left( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} \sqrt{1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2} \, dy \right) \, dx
\]

\[
= \frac{\pi c}{b} \int_{-a}^{a} b^2 \left( 1 - \left( \frac{x}{a} \right)^2 \right) \, dx
\]

\[
= \pi bc \int_{-a}^{a} 1 - \left( \frac{x}{a} \right)^2 \, dx
\]

\[
= \pi bc \left[ x - \frac{x^3}{3a^2} \right]_{-a}^{a}
\]

\[
= \pi bc \left( 2a - 2 \frac{a^3}{3a^2} \right)
\]

\[
= \frac{4\pi}{3} abc.
\]