MINIMAL SURFACES

Definition 0.1. We say that $S \subset \mathbb{R}^3$ is a minimal surface if it is a critical point for area.

We consider a particular class of minimal surfaces, minimal graphs, in what follows.

Let $u(x, y)$ be a graph of a surface $S \subset \mathbb{R}^3$ with $\Pi(S) = R$ and $u \in C^2(R)$. We know $\text{Area}(S) = \int \int_R \sqrt{1 + |\nabla u|^2} \, dx \, dy$. Now we determine what it means for $S$ to be a critical point for area. Consider any $v : R \to \mathbb{R}$ such that $v$ is continuously differentiable and $v = 0$ on $\partial R$. Then the function $u_t = u + tv : R \to \mathbb{R}$ and $u_t(\partial R) = \partial S$ for all $t$. Denote $S_t = u_t(R)$. We say $S$ is a critical point for area if

$$\frac{d}{dt}|_{t=0}\text{Area}(S_t) = 0.$$ 

Thus $S$ is a critical point for area iff

$$\frac{d}{dt}|_{t=0} \int \int_R \sqrt{1 + |\nabla u_t|^2} \, dx \, dy = 0.$$ 

But notice that $\nabla u_t = \nabla u + t\nabla v$ so $|\nabla u_t|^2 = |\nabla u|^2 + 2t\langle \nabla u, \nabla v \rangle + t^2|\nabla v|^2$. So

$$\frac{d}{dt} \sqrt{1 + |\nabla u_t|^2} = \frac{\langle \nabla u, \nabla v \rangle + t|\nabla v|^2}{\sqrt{1 + |\nabla u_t|^2}}.$$ 

Evaluating at $t = 0$ we get

$$\frac{\langle \nabla u, \nabla v \rangle}{\sqrt{1 + |\nabla u|^2}}.$$ 

Now we can interchange the limit and the integral because $v$ has continuous derivatives on $R$ and thus as $t \to 0$, $\nabla u_t \to \nabla u$ uniformly on $R$. Thus $S$ is a critical point for area if and only if for all $v \in C^1_0(R)$,

$$\int \int_R \frac{\langle \nabla u, \nabla v \rangle}{\sqrt{1 + |\nabla u|^2}} \, dx \, dy = 0.$$ 

Now recall

$$\int_{\partial R} F \cdot nds = \int \int_R (\partial F_1/\partial x + \partial F_2/\partial y) \, dy \, dx = \int \int_R \text{div}(F) \, dx \, dy.$$
where $n$ is the normal to the boundary of $\partial R$. Set $F = v \nabla u / \sqrt{1 + |\nabla u|^2}$; then $\int_{\partial R} F \cdot nds = 0$ (since $v \equiv 0$ on the boundary). Now, we compute $\text{div}(F)$:

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \frac{u_x}{\sqrt{1 + |\nabla u|^2}} + \frac{u_y}{\sqrt{1 + |\nabla u|^2}} + v \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

or

$$\text{div}(F) = \frac{\langle \nabla u, \nabla v \rangle}{\sqrt{1 + |\nabla u|^2}} + \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

Using (1) and (2) we see for all $v \in C^1_0(R)$,

$$0 = \int \int_R v \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) dxdy.$$

**Theorem 0.2.** Let $u(x, y)$ be a graph of a surface $S \subset \mathbb{R}^3$ with $\Pi(S) = R$ and $u \in C^2(S)$. Then $S$ is a minimal surface if and only if

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

**Proof.** Most of our work is already done. We know that $\int \int_R v \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) dxdy = 0$ for all $v \in C^1$ with $v = 0$ on the boundary of $R$. Now suppose there exists $(x', y') \in R$ such that

$$\text{div} \left( \frac{\nabla u(x', y')}{\sqrt{1 + |\nabla u(x', y')|^2}} \right) > 0.$$

Since $u \in C^2(R)$, it follows that there exists a neighborhood of $(x', y')$, $U \subset R$, such that $\text{div} \left( \frac{\nabla u(x, y)}{\sqrt{1 + |\nabla u(x, y)|^2}} \right) > 0$ for all $(x, y) \in U$. Now choose $v \in C^1(R)$ such that $v = 0$ on $R \setminus U$ and $v > 0$ in $U$. But then

$$\int \int_R v \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) dxdy = \int \int_U v \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) dxdy > 0$$

which provides a contradiction. \qed
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