Rough Solutions for PSet 12

1. (12.10:7) We first parameterize the sphere of radius $a$ by $r(\phi, \theta) = (a \cos \theta \sin \phi, a \sin \theta \sin \phi, a \cos \phi)$ where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. (We write $\phi$ first so that the normal points outward.) Then the integrals become

$$\int \int_S xy \, dy \wedge dz = \int_0^\pi \int_0^{2\pi} a^2 \sin^2 \phi \cos \theta \sin \theta (a^2 \cos \theta \sin^2 \phi) \, d\theta \, d\phi,$$

$$\int \int_S yz \, dz \wedge dx = \int_0^\pi \int_0^{2\pi} a^2 \sin \theta \sin \phi \cos \phi (a^2 \sin \theta \sin^2 \phi) \, d\theta \, d\phi,$$

$$\int \int_S x^2 \, dx \wedge dy = \int_0^\pi \int_0^{2\pi} a^2 \cos^2 \theta \sin^2 \phi (a^2 \cos \phi \sin \phi) \, d\theta \, d\phi.$$

From here the work is standard. Notice the second and third integral can be combined and simplified.

2. (12.10:12) We want to evaluate $\int \int_S F \cdot ndS$ and here $n = (x, y, z)$ since $S$ is a hemisphere. Thus $F \cdot n = x^2 - 2xy - y^2 + z^2$. As with the problem above, we parameterize the hemisphere by $r(\phi, \theta)$ where $0 \leq \phi \leq \pi/2$ and $\theta \in [0, 2\pi]$. So the problem is to evaluate

$$\int_0^{\pi/2} \int_0^{2\pi} (\cos^2 \theta \sin^2 \phi - 2 \cos \theta \sin \theta \sin^2 \phi - \sin^2 \theta \sin^2 \phi + \cos^2 \phi) \sin \phi \, d\theta \, d\phi.$$

The first three terms integrate to zero in $\theta$ so the only term that carries through to the second integral is $2\pi \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi = 2\pi/3$.

3. (12.10:13) There’s actually nothing to add in. This is because $F \cdot n$ on the disk is zero. ($z = 0$ on the $x, y$-plane and $n = (0, 0, -1)$). The book apparently has a different answer. So let’s just check this by using the divergence theorem. That is

$$\int \int \int_V \text{div}(F) \, dx \, dy \, dz = \int \int \int_V 1 \, dx \, dy \, dz = \text{Vol}(V)$$

where $V$ is the upper solid hemisphere. Thus, $\text{Vol}(V) = 2\pi/3$. But since

$$\int \int \int_V \text{div}(F) \, dx \, dy \, dz = \int \int_{S_{\text{cap}}} F \cdot n \, dS_{\text{cap}} + \int \int_{S_{\text{disk}}} F \cdot n \, dS_{\text{disk}}$$

we see the second term can contribute nothing.
4. (12.13:3) The line integral of interest will be over the square with side length 2, with two sides on the $x$- and $y$-axes. Before we write out all of the details, notice that on the $x, y$-plane, $F(x, y, 0) = (y, 0, 0)$ so the only parts of the line integral that matter are the parts parallel to the $x$-axis (do $dx \neq 0$). Thus, the problem reduces to finding

$$\int_0^2 F(t, 0, 0) \cdot (1, 0, 0) \, dt - \int_0^2 F(t, 2, 0) \cdot (1, 0, 0) \, dt.$$ 

The first integral is zero since $F \equiv 0$ on that curve. The second becomes

$$- \int_0^2 2 \, dt = -4.$$

5. (12.13:11) It is enough to show that $\nabla \times (a \times r) = 2a$. Then we can use Stokes’ Theorem:

$$\int_{\partial S} (a \times r) \cdot d\gamma = \int_S \nabla \times (a \times r) \cdot n \, dS.$$ 

But this is just a straightforward calculation. Let $a = (a_1, a_2, a_3)$ and $r = (x, y, z)$. Then

$$a \times r = (a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x).$$

Taking

$$\text{curl}(a \times r) = (a_1 + a_1, a_2 + a_2, a_3 + a_3)$$

gives the result.
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