Solutions for PSet 8

1. (10.5:11) Parameterize the sides of the square \( C \) by maps \( s_i : [0, 1] \to \mathbb{R}^2 \) by

\[
\begin{align*}
s_1(t) &= (1 - t, t); \\
s_2(t) &= (-t, 1 - t); \\
s_3(t) &= (t - 1, -t); \\
s_4(t) &= (t, t - 1).
\end{align*}
\]

With this parametrization:

\[
\int_C \frac{dx + dy}{|x| + |y|} = \int_0^1 \frac{-1 + 1}{(1 - t) + t} dt + \int_0^1 \frac{-1 - 1}{t + (1 - t)} dt + \int_0^1 \frac{1 - 1}{(1 - t) + t} dt + \int_0^1 \frac{1 + 1}{t + (1 - t)} dt
\]

The first and the third summands are 0, and the second and fourth terms cancel each other, giving:

\[
\int_C \frac{dx + dy}{|x| + |y|} = 0
\]

2. (10.9:6) Writing the equation of the cylinder in complete square form:

\[
(x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}
\]

Thus looking from high above the \( xy \)-plane the picture looks like:

The parametrization of the cylinders’ intersection with the \( xy \)-plane is:

\[
\tilde{s}(t) = \left( \frac{a}{2} \cos t + \frac{a}{2}, \frac{a}{2} \sin t, 0 \right)
\]
We need to lift it up to sit on the sphere:

\[ s(t) = \left( \frac{a}{2} \cos t + \frac{a}{2}, \frac{a}{2} \sin t, z(t) \right), \]

where \( z(t) \geq 0 \) and

\[
\left( \frac{a}{2} \cos t + \frac{a}{2} \right)^2 + \left( \frac{a}{2} \sin t \right)^2 + z(t)^2 = a \left( \frac{a}{2} \cos t + \frac{a}{2} \right) + z(t)^2 = a^2
\]

This means, that

\[ z(t) = \frac{a}{\sqrt{2}} \sqrt{1 - \cos t} \]

Now

\[
\int_C (y^2, z^2, x^2) \cdot d(x, y, z) = \int_0^{2\pi} \frac{a^3}{8} \left( \sin^2 t, 2(1 - \cos t), (\cos t + 1)^2 \right) \cdot \left( -\sin t, \cos t, \frac{\sin t}{\sqrt{2(1 - \cos t)}} \right) \, dt
\]

\[
= \frac{a^3}{8} \int_0^{2\pi} \left( -\sin^2 t + 2 \cos t(1 - \cos t) + \frac{\sin t(\cos t + 1)^2}{\sqrt{2(1 - \cos t)}} \right) \, dt
\]

\[
= -\frac{a^3}{8} \int_0^{2\pi} \sin^3 t \, dt + \frac{a^3}{4} \int_0^{2\pi} \cos t(1 - \cos t) \, dt + \frac{a^3}{8} \int_0^{2\pi} \frac{\sin t(\cos t + 1)^2}{\sqrt{2(1 - \cos t)}} \, dt
\]

Computing each of the integrals separately we get:

\[
= 0 + \frac{a^3}{4} \pi + 0 = \frac{a^3 \pi}{4}
\]

3. (C34:3) As per the question, \( f(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \). Therefore we can write

\[ \phi(x, y) = \int_C \frac{1}{x^2 + y^2} (-y, x) \cdot d(x, y) \]

As suggested in the exercise we will compute the integral along a specific path starting at \((1, 0)\). For given \((x, y)\) we can parameterize the path in two parts with \( s_1 : [1, x] \to \mathbb{R}^2 \) and \( s_2 : [0, y] \to \mathbb{R}^2 \). (Here an interval \([a, b]\) is understood as \([b, a]\) if \( a > b \).)

\[
s_1(t) = (t, 0)
\]

\[
s_2(t) = (x, t)
\]

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With these notations:

\[
\phi(x, y) = \int_C \frac{1}{x^2 + y^2} (-y, x) \cdot d(x, y)
\]

\[
= \int_1^x -\frac{0}{t^2} dt + \int_0^y \frac{x}{x^2 + t^2} dt = \arctan \frac{y}{x}
\]

Finally, we can check that this is indeed the potential function for \(f(x, y)\):

\[
\nabla \phi(x, y) = \frac{1}{x^2 + y^2} (-y, x) = f(x, y).
\]

4. (10.18:13) Note, that the function is not necessarily well defined in \((0, 0)\). Thus we will fix our basepoint at \((1, 0)\). Then given a point \(r(\cos \vartheta, \sin \vartheta) \in \mathbb{R}^2\), then an obvious path from \((1, 0)\) to \(r(\cos \vartheta, \sin \vartheta)\) can be parametrized by \(s_1 : [0, \vartheta] \to \mathbb{R}^2\) and \(s_2 : [1, r] \to \mathbb{R}^2\) with

\[
s_1(t) = (\cos t, \sin t) \\
s_2(t) = t(\cos \vartheta, \sin \vartheta)
\]

For \(n \neq -1\)

\[
\phi(r(\cos \vartheta, \sin \vartheta)) = \int_0^\vartheta a1^n(\cos t, \sin t) \cdot (-\sin t, \cos t) dt + \int_1^r at^n(\cos \vartheta, \sin \vartheta) \cdot (\cos \vartheta, \sin \vartheta) dt
\]

\[
= 0 + a \int_1^r t^n dt = \frac{ar^{n+1}}{n+1} - \frac{a}{n+1}
\]

Checking that it is a potential function:

\[
\nabla \frac{ar^{n+1}}{n+1} = ar^n(\cos \vartheta, \sin \vartheta)
\]

For \(n = -1\) we have

\[
\psi(r(\cos \vartheta, \sin \vartheta)) = \int_1^r \frac{a}{t} (\cos \vartheta, \sin \vartheta) \cdot (\cos \vartheta, \sin \vartheta) dt = a \int_1^r \frac{1}{t} dt = a \log r
\]

Again checking that this is indeed a potential function:

\[
\nabla \psi(r(\cos \vartheta, \sin \vartheta)) = \frac{a}{r}(\cos \vartheta, \sin \vartheta).
\]
5. (10.18:17,18) In this exercise

\[ f(x, y) = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \]

10.18:17 We have computed on the recitation that

\[ D_1 f_2(x, y) = D_2 f_1(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} \]

10.18:18 (Compare the results with 3)

(a) We will consider the 3 cases one by one. First, for \( x = 0 \) we have, by definition, \( \theta = \pi/2 \). Now when \( x \neq 0 \)

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{r \sin \theta}{r \cos \theta} = \frac{y}{x}, \]

and

\[ \arctan \frac{y}{x} = \arctan \frac{-y}{-x} = \phi \in (-\pi/2, \pi/2). \]

For \( x > 0 \), \( -\pi/2 < \theta = \phi < \pi/2 \) and this corresponds directly with the definition of the arctan function.
For \( x < 0 \), it turns out that \( \theta = \phi + \pi \) because the angle between \( (x, y) \) and \( (-x, -y) \) is precisely \( \pi \).

(b) Using the derivation rule for the inverse function. If \( x > 0 \)

\[ \frac{\partial \theta}{\partial x} (x, y) = \frac{\partial}{\partial x} \arctan \frac{y}{x} \]

\[ = -\frac{y}{x^2 + (\frac{y}{x})^2} = -\frac{y}{x^2 + y^2} \]

\[ \frac{\partial \theta}{\partial y} (x, y) = \frac{\partial}{\partial y} \arctan \frac{y}{x} \]

\[ = \frac{1}{x} \frac{1}{1 + (\frac{y}{x})^2} = \frac{x}{x^2 + y^2} \]

Similar argument works for \( x < 0 \) case. For \( x = 0 \) one computes the left and right derivatives, and see that they are both equal to:

\[ \frac{\partial \theta}{\partial x} (0, y) = -\frac{1}{y} \]
and

\[ \frac{\partial \theta}{\partial y}(0, y) = 0. \]

Hence for all \((x, y)\), the relations in the exercise for \(\frac{\partial \theta}{\partial x}\) and \(\frac{\partial \theta}{\partial y}\) hold. This proves that \(\theta\) is a potential function for \(f\) on the set \(T\).