Solutions for PSet 9

1. (11.9:8) Using Fubini’s Theorem (we assumed that the double integral exists):

\[
\int \int_{[0,t] \times [1,t]} \frac{e^{tx}}{y^3} \, dx \, dy = \int_1^t \left( \int_0^t \frac{e^{tx}}{y^3} \, dx \right) \, dy = \int_1^t e^{tx} - \frac{1}{ty^2} \, dy = \left[ \frac{t^2 e^y}{y^3} + \frac{1}{ty} \right]_{y=0}^{y=1} = \frac{1}{t^2} - \frac{1}{t} - \frac{1}{t^3} e^t + \frac{1}{t^3} e^{t^2}
\]

2. (11.15:2)

\[
\int \int_S (1 + x) \sin y \, dx \, dy = \int_0^1 \left( \int_0^{1+x} (1 + x) \sin y \, dy \right) \, dx = \left[ \frac{x}{2} (x + 2) - (x + 1) \sin(x + 1) - \cos(x + 1) \right]_0^1 = \frac{3}{2} + \cos 1 + \sin 1 - \cos 2 - 2 \sin 2
\]

3. (11.15:6) The volume can be computed as the double integral of the function \( f(x, y) = \frac{6 - x - 2y}{3} \) over region \( S = \{(x, y) | 0 \leq x \leq 6, 0 \leq y \leq (6 - x)/2 \} \):

\[
\int \int_S \frac{6 - x - 2y}{3} \, dy \, dx = \int_0^6 \left( \int_0^{6-x/2} \frac{6-x-2y}{3} \, dy \right) \, dx = \left[ \frac{(6-x)^2}{12} \right] \left[ -\frac{(6-x)^3}{36} \right]_0^6 = 6
\]

4. (11.15:13) The domain we integrate over is given as

\[ S = \{-6 \leq x \leq 2, \quad \frac{x^2 - 4}{4} \leq y \leq 2 - x \} \]
Observe the points of intersection of the two functions of \( x \) are at \((-6, 8)\) and \((2, 0)\). Integrating in \( x \) first will require dividing the domain into two regions, as on \( 0 \leq y \leq 8, -\sqrt{4+4y} \leq x \leq 2-y \) while on \(-1 \leq y \leq 0 \) we see \(-\sqrt{4+4y} \leq x \leq \sqrt{4+4y} \).

Therefore we can evaluate our integral
\[
\int_{6}^{2} \int_{\frac{x^2}{4}}^{2-x} f(x, y) \, dy \, dx = \int_{-1}^{0} \int_{-\sqrt{4y+4}}^{\sqrt{4y+4}} f(x, y) \, dx \, dy + \int_{0}^{8} \int_{-\sqrt{4y+4}}^{2-y} f(x, y) \, dx \, dy
\]

5. (11.18:10) Place the coordinate system so that the sides of the rectangle become parallel to the axis and \( A = (0, 0) \), \( B = (0, b) \), \( C = (a, b) \) and \( D = (a, 0) \). The side \( AB \) then is along the \( y \) axis and the side \( AD \) is along the \( x \) axis. The rectangle can be described as \( Q = \{0 \leq x \leq a, 0 \leq y \leq b\} \). The distances of any point \((x, y)\) from segment \( AB \) and \( AD \) are \( x \) and \( y \) respectively. Thus, density \( f(x, y) \) and mass \( m(Q) \) can be defined as:
\[
f(x, y) = x \times y
\]
\[
m(Q) = \int \int_{Q} f(x, y) \, dy \, dx = \left(\frac{ab}{2}\right)^{2}
\]

Then the coordinates of the center of mass can be computed as:
\[
\bar{x} = \frac{1}{m(Q)} \int \int_{Q} x(xy) \, dy \, dx = \frac{2}{3}a
\]
\[
\bar{y} = \frac{1}{m(Q)} \int \int_{Q} y(xy) \, dy \, dx = \frac{2}{3}b
\]

6. Let \( f_S, f_R \) represent the density functions for \( S, R \) respectively. We define
\[
f_{R \cup S}(x) = \begin{cases} f_R(x) & \text{if } x \in R \\ f_S(x) & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}
\]

Then
\[
\overline{x_T} = \frac{\int \int_{R \cup S} x f_{R \cup S} \, dx \, dy}{\int \int_{R \cup S} f_{R \cup S} \, dx \, dy} = \frac{\int \int_{R} x f_R \, dx \, dy + \int \int_{S} x f_S \, dx \, dy}{\int \int_{R} f_R \, dx \, dy + \int \int_{S} f_S \, dx \, dy}.
\]
Now observe that \( \int_R x f_R \, dx \, dy = \bar{x}_R \int_R f_R \, dx \, dy = x_R m(R) \) and \( \int_S x f_S \, dx \, dy = \bar{x}_S \int_S f_S \, dx \, dy = x_S m(S) \). Thus

\[
\bar{x}_T = \frac{\bar{x}_R m(R) + \bar{x}_S m(S)}{m(R) + m(S)}.
\]

A similar argument works for \( \bar{y}_T \) and the result follows immediately.