Solutions for PSet 1

1. (1.10:22)

(a) Let \( S = \{x_1, \ldots, x_k\} \subset V \). As \( L(S) = \text{span}(S) \), we can write:

\[
L(S) = \{ y : y \in V \text{ where } y = \sum_{i=1}^{k} c_i x_i \} \text{ is scalar}
\]

For \( c_j = 1, c_i = 0, i \neq j \), we have \( y = \sum_i c_i x_i = x_j \in L(S) \). Thus \( x_j \in S \) implies that \( x_j \in L(S) \) and \( S \subseteq L(S) \).

(b) As \( T \) is a subspace of linear space \( V \), \( T \) is a non-empty subset of \( V \) and \( T \) satisfies all closure axioms. Since \( S \subseteq T \), we know (using the notation above) that \( \{x_1, \ldots, x_k\} \subseteq T \). Now let \( y \in L(S) \). Then by definition there exist \( c_i \in \mathbb{R} \), for \( i = 1, \ldots, k \), such that \( y = \sum_i c_i x_i \). By the closure axioms, \( \sum_i c_i x_i \in T \) and thus \( L(S) \subseteq T \).

(c) Since \( L(S) \) is a subspace of \( V \), one direction is obvious.

Now, suppose by contradiction that \( S \) is a subspace of \( V \) but \( S \neq L(S) \).

Since \( S \subseteq L(S) \), this implies there exists \( y \in L(S) - S \). As \( y \in L(S) \), there exist \( c_i \in \mathbb{R} \), \( i = 1, \ldots, k \), such that \( y = \sum_i c_i x_i \). As \( S \) is a subset and thus closed under addition and scalar multiplication, \( y \in S \). This implies a contradiction and proves the result.

(d) Assume \( S = \{x_1, \ldots, x_k\}, T = \{x_1, \ldots, x_n\} \) where \( n \geq k \). Let \( y \in L(S) \).

Then \( y = \sum_{i=1}^{k} c_i x_i \) for some \( c_i \in \mathbb{R} \). For \( c_j = 0 \) for all \( j = k+1, \ldots, n \), \( y = \sum_{i=1}^{k} c_i x_i + \sum_{j=k+1}^{n} c_j x_j \). Thus, \( y \in L(T) \).

(e) As \( S \) and \( T \) are subspaces of \( V \), they are both closed under addition and scalar multiplication. Let \( x, y \in S \cap T \) and \( c \in \mathbb{R} \). As \( cx + y \in S \) and \( cx + y \in T \) we see \( cx + y \in S \cap T \). Thus \( S \cap T \) is closed under addition and multiplication. Therefore \( S \cap T \) is a subspace of \( V \).

(f) Assume \( S = \{x_1, \ldots, x_k\}, T = \{y_1, \ldots, y_n\} \). Let \( z \in L(S \cap T) \). Then there exist \( c_j \in \mathbb{R} \) and \( z_j \in S \cap T \) such that \( z = \sum_j c_j z_j \). Since \( z_j \in S \cap T \), \( \sum_j c_j z_j \in L(S), L(T) \). Thus, \( z \in L(S) \cap L(T) \).

(g) Let \( S = \{v_1, v_2\}, T = \{v_3, v_4\} \) where \( v_i \in \mathbb{R}^3 \) are each vectors such that \( v_3, v_4 \notin L(S) \) and \( v_1, v_2 \notin L(T) \). We can further choose these vectors such that \( L(S) \) and \( L(T) \) are both planes in \( \mathbb{R}^3 \) by making sure each pair of vectors is linearly independent. By construction, \( S \cap T = \emptyset \) but \( L(S) \cap L(T) \) is a line in \( \mathbb{R}^3 \). So \( L(S) \cap L(T) \neq L(S \cap T) \).
2. (1.13:11) In the linear space of all real polynomials, define \( (f, g) = \int_{0}^{\infty} e^{-t} f(t) g(t) \, dt \).

(a) Let \( f, g \) be polynomials. Then \( f g = \sum_{i=0}^{n} a_i x^i \) for some \( n \in \mathbb{N} \), \( a_i \in \mathbb{R} \).

By definition,
\[
(f, g) = \int_{0}^{\infty} \sum_{i=0}^{n} e^{-t} a_i t^i \, dt.
\]

Using integration by parts, we see that for any fixed \( n \in \mathbb{N} \),
\[
\int_{0}^{\infty} t^n e^{-t} \, dt = -t^n e^{-t} \bigg|_{0}^{\infty} + \int_{0}^{\infty} nt^{n-1} e^{-t} \, dt = \int_{0}^{\infty} nt^{n-1} e^{-t} \, dt.
\]

Iteratively integrating by parts \( n \) times, we see
\[
\int_{0}^{\infty} t^n e^{-t} \, dt = n! \int_{0}^{\infty} e^{-t} \, dt = n!.
\]
(To be truly thorough, one should prove this by induction but we leave that to you!)

Thus, for \( f g = \sum_{i=0}^{n} a_i x^i \),
\[
(f, g) = \sum_{i=0}^{n} i! a_i < \infty.
\]

(b)
\[
(x_n, x_m) = \int_{0}^{\infty} e^{-t} t^n t^m \, dt = \int_{0}^{\infty} e^{-t} t^{m+n} \, dt
\]
\[
= (m + n) \int_{0}^{\infty} e^{-t} t^{m+n-1} \, dt
\]
\[
= (m + n)(m + n - 1) \int_{0}^{\infty} e^{-t} t^{m+n-2} \, dt \quad \text{(by iteratively integrating by parts)}
\]
\[
= (m + n)(m + n - 1) \cdots 1 \left[ \int_{0}^{\infty} e^{-t} \, dt \right]
\]
\[
= (m + n)(m + n - 1) \cdot 1 \cdot 1 = (m + n)!
\]
(c) If \( g(t) \) orthogonal to \( f(t) \), then:

\[
(f,g) = \int_{0}^{\infty} e^{-t} (a + bt)(1 + t) \, dt \\
= \int_{0}^{\infty} ae^{-t} \, dt + (a + b) \int_{0}^{\infty} te^{-t} \, dt + b \int_{0}^{\infty} t^2 e^{-t} \, dt = 0
\]

\[
\Rightarrow a + a + b + 2b = 2a + 3b = 0
\]

This means that \( 2a = -3b \) or polynomials \( g(t) = a \left( 1 - \frac{2}{3}t \right) \) satisfy the requirement of orthogonality to \( f(t) = 1 + t \).

3. (2.4:29) Let \( V \) denote the linear space of all real functions continuous on the interval \([-\pi, \pi]\). Let \( S \) be that subset of \( V \) consisting of all \( f \) satisfying:

\[
\int_{-\pi}^{\pi} f(t) \, dt = \int_{-\pi}^{\pi} f(t) \cos t \, dt = \int_{-\pi}^{\pi} f(t) \sin t \, dt.
\]

(a) By definition, \( S \subseteq V \). As integration is a linear operation, it can be shown that for \( f_1, f_2 \in S \) and \( a \in \mathbb{R} \),

\[
\int_{-\pi}^{\pi} f_1(t) + f_2(t) \, dt = \int_{-\pi}^{\pi} (f_1(t) + f_2(t)) \cos t \, dt = \int_{-\pi}^{\pi} (f_1(t) + f_2(t)) \sin t \, dt
\]

and

\[
\int_{-\pi}^{\pi} a f(t) \, dt = \int_{-\pi}^{\pi} a f(t) \cos t \, dt = \int_{-\pi}^{\pi} a f(t) \sin t \, dt.
\]

Thus, \( S \) is closed under addition and scalar multiplication.

(b) \( S \) contains the functions \( f(x) \) defined above if those functions are real and are a part of \( V \). Thus we have to show that \( f(x) = \cos(nx) \) and \( f(x) = \sin(nx) \) satisfy the integral equations defining \( V \). Start with \( f(x) = \cos(nx) \):

\[
\int_{-\pi}^{\pi} \cos(nt) \, dt = \frac{1}{n} [\sin(\pi) - \sin(-\pi)] = 0
\]

\[
\int_{-\pi}^{\pi} \cos(nt) \cos(t) \, dt = 0.5 \int_{-\pi}^{\pi} [\cos(nt + t) + \cos(nt - t)] \, dt = 0
\]

\[
\int_{-\pi}^{\pi} \cos(nt) \sin(t) \, dt = 0.5 \int_{-\pi}^{\pi} [\sin(nt + t) - \sin(nt - t)] \, dt
\]

\[
= 0.5 \cos(-(n + 1)x) - 0.5 \cos((n + 1)x)
\]

\[
+ 0.5 \cos((n - 1)x) - 0.5 \cos(-(n - 1)x) = 0
\]

(for both even and odd \( n \))
A similar derivation makes the case for \( f(x) = \sin(nx) \).

(c) \( S \) is infinite dimensional if its basis has an infinite number of independent elements. We can prove it is infinite dimensional by proving it is not finite dimensional. As \( f_n(x) = \cos(nx) \), \( f_n(x) \) is orthogonal to \( f_m(x) \) for all \( n > 2 \neq m > 2 \). Therefore there is no finite basis set of independent elements that can span \( S \).

(d) Using trigonometric identities, observe that for \( g(x) \in T(V) \) one has

\[
g(x) = \int_{-\pi}^{\pi} f(t)dt + \cos(x) \int_{-\pi}^{\pi} \cos(t)f(t)dt + \sin(x) \int_{-\pi}^{\pi} \sin(t)f(t)dt.
\]

Thus, \( T(V) \) is three dimensional with basis \( \{1, \cos(x), \sin(x)\} \). (Note that since \( f \in V \), the three integrals are all elements of \( \mathbb{R} \).)

(e) Based on the identity shown in the previous part of the problem, \( g = T(f) = 0 \) if and only if the three integrals are all zero. Thus, \( N(T) \) is precisely equal to the subspace \( S \).

(f) Using the hint, observe that \( f(x) = c_1 + c_2 \cos x + c_3 \sin x \) for some \( c_1, c_2, c_3 \in \mathbb{R} \). Now evaluating the three integrals that describe \( T(f) \) we observe

\[
T(f) = cf(x) = 2\pi c_1 + \pi c_2 \cos x + \pi c_3 \sin x.
\]

Thus, if \( c_1 = 0 \) then \( f(x) = c_2 \cos x + c_3 \sin x \) and \( c = \pi \) (here \( c_2, c_3 \) are arbitrary real numbers). If \( c_1 \neq 0 \), then \( c_2 = c_3 = 0 \), \( f(x) \) is a constant function and \( c = 2\pi \).

(g) Let

\[
f_j(x) = \begin{cases} 
1 & \text{if } x \in [-1/j, 1/j] \\
0 & \text{otherwise}
\end{cases}
\]

Then \( f_j \to 0 \) strongly in \( L^2 \) as

\[
\int_{\mathbb{R}} |f_j - 0|^2 dx = 4/j^2 \to 0.
\]

Observe, however, that \( f_j \to f_\infty \) pointwise, where

\[
f_\infty(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{otherwise}
\end{cases}
\]

Notice here that \( f_j \) actually also converges strongly to \( f_\infty \) in \( L^2 \), so though we’ve found a solution it might not be fully satisfying. A harder question to solve would be the following: Find a sequence of functions \( f_j \) such that NO subsequence of \( f_j \) converges pointwise to a function but \( f_j \) still converges strongly in \( L^2 \) to a function.