Dot Product

The dot product is one way of combining ("multiplying") two vectors. The output is a scalar (a number). It is called the dot product because the symbol used is a dot. Because
the dot product results in a scalar it, is also called the scalar product.
As with most things in 18.02, we have a geometric and algebraic view of dot product.

Algebraic definition (for 2D vectors):
If \( \mathbf{A} = (a_1, a_2) \) and \( \mathbf{B} = (b_1, b_2) \) then
\[
\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2.
\]

Example: \( \langle 6, 5 \rangle \cdot \langle 1, 2 \rangle = 6 \cdot 1 + 5 \cdot 2 = 16. \)

Geometric view:
The figure below shows \( \mathbf{A}, \mathbf{B} \) with the angle \( \theta \) between them. We get
\[
\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta
\]

Showing the two views (algebraic and geometric) are the same requires the law of cosines
\[
|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos \theta
\]
\[
\Rightarrow (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - ((a_1 - b_1)^2 + (a_2 - b_2)^2) = 2|\mathbf{A}||\mathbf{B}| \cos \theta
\]
\[
\Rightarrow a_1 b_1 + a_2 b_2 = |\mathbf{A}||\mathbf{B}| \cos \theta.
\]
Since \( \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2 \), we have shown \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta. \)

From the algebraic definition of dot product we easily get the the following algebraic law
\[
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}.
\]

Example: Find the dot product of \( \mathbf{A} \) and \( \mathbf{B} \).
i) \( |\mathbf{A}| = 2, \ |\mathbf{B}| = 5, \ \theta = \pi/4. \)
Answer: (draw the picture yourself) \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta = 10\sqrt{2}/2 = 5\sqrt{2}. \)

ii) \( \mathbf{A} = \mathbf{i} + 2\mathbf{j}, \ \mathbf{B} = 3\mathbf{i} + 4\mathbf{j}. \)
Answer: \( \mathbf{A} \cdot \mathbf{B} = 1 \cdot 3 + 2 \cdot 4 = 11. \)

Three dimensional vectors
The dot product works the same in 3D as in 2D. If \( \mathbf{A} = \langle a_1, a_2, a_3 \rangle \) and \( \mathbf{B} = \langle b_1, b_2, b_3 \rangle \) then
\[
\mathbf{A} \cdot \mathbf{B} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3.
\]

The geometric view is identical and the same proof shows
\[
\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta
\]
Example:
Show $A = (4, 3, 6), B = (-2, 0, 8), C = (1, 5, 0)$ are the vertices of a right triangle.

Answer: Two legs of the triangle are $\overrightarrow{AC} = (-3, 2, -6)$ and $\overrightarrow{AB} = (-6, -3, 2) \Rightarrow \overrightarrow{AC} \cdot \overrightarrow{AB} = 18 - 6 - 12 = 0$. The geometric view of dot product implies the angle between the legs is $\pi/2$ (i.e $\cos \theta = 0$).

![Diagram of vectors A, B, and C forming a right triangle with dot product calculations]

Definition of the term orthogonal and the test for orthogonality
When two vectors are perpendicular to each other we say they are orthogonal.
As seen in the example, since $\cos(\pi/2) = 0$, the dot product gives a test for orthogonality between vectors:

$$A \perp B \iff A \cdot B = 0.$$  

Dot product and length
Both the algebraic and geometric formulas for dot product show it is intimately connected to length. In fact, they show for a vector $\mathbf{A}$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2.$$  

Let’s show this using both views.
Algebraically: suppose $\mathbf{A} = (a_1, a_2, a_3)$ then

$$\mathbf{A} \cdot \mathbf{A} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = |\mathbf{A}|^2.$$  

Geometrically: the angle $\theta$ between $\mathbf{A}$ and itself is 0. Therefore,

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}| \cos \theta = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2.$$  

As promised both views give the formula.