Meaning of matrix multiplication

In these examples we will explore the effect of matrix multiplication on the $xy$-plane.

**Example 1:** The matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$ transforms the unit square into a parallelogram as follows.

The unit square has sides $i$ and $j$. In order multiply a matrix times a vector we write them as column vectors. For example, $i = \langle 1, 0 \rangle$, $j = \langle 0, 1 \rangle$ and $v = \langle a_1, a_2 \rangle$ are written

\[
i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}
\]

The matrix multiplication then becomes

\[
Ai = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}; \quad Aj = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\]

We think of the all the points in the square as the endpoints of origin vectors. If we multiply $A$ by all of these vectors we get the following picture.

The square is mapped to the parallelogram. We know that the area of the parallelogram is $|A| = 11$. (Think about the $2 \times 2$ determinant you would use to compute the area of the parallelogram.)