Matrices 1. Matrix Algebra

Matrix algebra.

Previously we calculated the determinants of square arrays of numbers. Such arrays are important in mathematics and its applications; they are called matrices. In general, they need not be square, only rectangular.

A rectangular array of numbers having $m$ rows and $n$ columns is called an $m \times n$ matrix. The number in the $i$-th row and $j$-th column (where $1 \leq i \leq m$, $1 \leq j \leq n$) is called the $ij$-entry, and denoted $a_{ij}$; the matrix itself is denoted by $A$, or sometimes by $(a_{ij})$.

Two matrices of the same size are equal if corresponding entries are equal.

Two special kinds of matrices are the row-vectors: the $1 \times n$ matrices $(a_1, a_2, \ldots, a_n)$; and the column vectors: the $m \times 1$ matrices consisting of a column of $m$ numbers.

From now on, row-vectors or column-vectors will be indicated by boldface small letters; when writing them by hand, put an arrow over the symbol.

Matrix operations

There are four basic operations which produce new matrices from old.

1. **Scalar multiplication**: Multiply each entry by $c : cA = (ca_{ij})$

2. **Matrix addition**: Add the corresponding entries: $A + B = (a_{ij} + b_{ij})$; the two matrices must have the same number of rows and the same number of columns.

3. **Transposition**: The transpose of the $m \times n$ matrix $A$ is the $n \times m$ matrix obtained by making the rows of $A$ the columns of the new matrix. Common notations for the transpose are $A^T$ and $A'$; using the first we can write its definition as $A^T = (a_{ji})$.

   If the matrix $A$ is square, you can think of $A^T$ as the matrix obtained by flipping $A$ over around its main diagonal.

**Example 1.1** Let $A = \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ -2 & 3 \\ -1 & 0 \end{pmatrix}$. Find $A + B$, $A^T$, $2A - 3B$.

**Solution.** $A + B = \begin{pmatrix} 3 & 2 \\ -2 & 4 \\ -2 & 2 \end{pmatrix}$; $A^T = \begin{pmatrix} 2 & 0 & -1 \\ -3 & 1 & 2 \end{pmatrix}$; $2A - 3B = \begin{pmatrix} 4 & -6 \\ 0 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & -15 \\ 6 & -9 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -21 \\ 6 & -7 \\ 1 & 4 \end{pmatrix}$.

4. **Matrix multiplication** This is the most important operation. Schematically, we have

$$A \cdot B = C$$

$$m \times n \quad n \times p \quad m \times p$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$
The essential points are:
1. For the multiplication to be defined, \( A \) must have as many columns as \( B \) has rows;
2. The \( ij \)-th entry of the product matrix \( C \) is the dot product of the \( i \)-th row of \( A \) with the \( j \)-th column of \( B \).

**Example 1.2** \[
\begin{pmatrix}
2 \\
1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
2 & 4
\end{pmatrix} = \begin{pmatrix}
-2 + 4 - 2 = 0
\end{pmatrix};
\]
\[
\begin{pmatrix}
1 & 0 \\
2
\end{pmatrix}
\begin{pmatrix}
4 & 5 \\
8 & 10
\end{pmatrix} = \begin{pmatrix}
2 - 1 - 2 & 0 & 1 & 2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 3 - 2 - 6 \\
0 & 2 & 2 & 0
\end{pmatrix}
\]

The two most important types of multiplication, for multivariable calculus and differential equations, are:
1. \( AB \), where \( A \) and \( B \) are two square matrices of the same size — these can always be multiplied;
2. \( Ab \), where \( A \) is a square \( n \times n \) matrix, and \( b \) is a column \( n \)-vector.

**Laws and properties of matrix multiplication**

\( \text{M-1.} \quad A(B + C) = AB + AC; \quad (A + B)C = AC + BC \quad \text{distributive laws} \)

\( \text{M-2.} \quad (AB)C = A(BC); \quad (cA)B = c(AB). \quad \text{associative laws} \)

In both cases, the matrices must have compatible dimensions.

\( \text{M-3.} \quad \text{Let } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \text{then } AI = A \text{ and } IA = A \text{ for any } 3 \times 3 \text{ matrix.} \)

\( I \) is called the identity matrix of order 3. There is an analogously defined square identity matrix \( I_n \) of any order \( n \), obeying the same multiplication laws.

\( \text{M-4.} \quad \text{In general, for two square } n \times n \text{ matrices } A \text{ and } B, \quad AB \neq BA: \text{ matrix multiplication is not commutative.} \) (There are a few important exceptions, but they are very special — for example, the equality \( AI = IA \) where \( I \) is the identity matrix.)

\( \text{M-5.} \quad \text{For two square } n \times n \text{ matrices } A \text{ and } B, \text{ we have the determinant law: } \)

\[
|AB| = |A||B|, \quad \text{also written } \quad \det(AB) = \det(A)\det(B)
\]

For \( 2 \times 2 \) matrices, this can be verified by direct calculation, but this naive method is unsuitable for larger matrices; it’s better to use some theory. We will simply assume it in these notes; we will also assume the other results above (of which only the associative law \( \text{M-2} \) offers any difficulty in the proof).

\( \text{M-6.} \quad \text{A useful fact is this: matrix multiplication can be used to pick out a row or column of a given matrix: you multiply by a simple row or column vector to do this. Two examples} \)
should give the idea:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$
the second column

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$
the first row