18.02 Practice Exam 2 A

Problem 1. (10 points: 5, 5)
Let \( f(x, y) = xy - x^4 \).

a) Find the gradient of \( f \) at \( P : (1, 1) \).

b) Give an approximate formula telling how small changes \( \Delta x \) and \( \Delta y \) produce a small change \( \Delta w \) in the value of \( w = f(x, y) \) at the point \( (x, y) = (1, 1) \).

Problem 2. (20 points)
On the topographical map below, the level curves for the height function \( h(x, y) \) are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given.

a) Estimate to the nearest .1 the value at the point \( P \) of the directional derivative \( \frac{dh}{ds} \hat{u} \), where \( \hat{u} \) is the unit vector in the direction of \( \hat{i} + \hat{j} \).

b) Mark on the map a point \( Q \) at which \( h = 2200 \), \( \frac{\partial h}{\partial x} = 0 \) and \( \frac{\partial h}{\partial y} < 0 \). Estimate to the nearest .1 the value of \( \frac{\partial h}{\partial y} \) at \( Q \).

Problem 3. (10 points)
Find the equation of the tangent plane to the surface \( x^3y + z^2 = 3 \) at the point \((-1, 1, 2)\).
Problem 4. (20 points: 5,5,5,5)

A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P : (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. Which $P$ gives the box of greatest volume?

a) Show that the problem leads one to maximize $f(x, y) = xy - x^3 y - xy^3$, and write down the equations for the critical points of $f$.

b) Find a critical point of $f$ which lies in the first quadrant ($x > 0$, $y > 0$).

c) Determine the nature of this critical point by using the second derivative test.

d) Find the maximum of $f$ in the first quadrant (justify your answer).

Problem 5. (15 points)

In Problem 4 above, instead of substituting for $z$, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

a) Write down the Lagrange multiplier equations for this problem.

b) Solve the equations (still assuming $x > 0$, $y > 0$).

Problem 6. (10 points)

Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\partial w / \partial x$ and $\partial w / \partial y$ in terms of $x$, $y$, $f_u$ and $f_v$.

Problem 7. (15 points)

Suppose that $x^2 y + xz^2 = 5$, and let $w = x^3 y$. Express $\left( \frac{\partial w}{\partial z} \right)_y$ as a function of $x$, $y$, $z$, and evaluate it numerically when $(x, y, z) = (1, 1, 2)$. 