Partial derivatives

Let $w = f(x, y)$ be a function of two variables. Its graph is a surface in $xyz$-space, as pictured.

Fix a value $y = y_0$ and just let $x$ vary. You get a function of one variable,

(1) \[ w = f(x, y_0), \quad \text{the partial function for } y = y_0. \]

Its graph is a curve in the vertical plane $y = y_0$, whose slope at the point $P$ where $x = x_0$ is given by the derivative

(2) \[ \frac{d}{dx} f(x, y_0) \bigg|_{x_0}, \quad \text{or } \frac{\partial f}{\partial x} \bigg|_{(x_0, y_0)}. \]

We call (2) the partial derivative of $f$ with respect to $x$ at the point $(x_0, y_0)$; the right side of (2) is the standard notation for it. The partial derivative is just the ordinary derivative of the partial function — it is calculated by holding one variable fixed and differentiating with respect to the other variable. Other notations for this partial derivative are

\[
\begin{align*}
    f_x(x_0, y_0), & \quad \frac{\partial w}{\partial x} \bigg|_{(x_0, y_0)}, & \quad \left( \frac{\partial f}{\partial x} \right)_0, & \quad \left( \frac{\partial w}{\partial x} \right)_0;
\end{align*}
\]

the first is convenient for including the specific point; the second is common in science and engineering, where you are just dealing with relations between variables and don’t mention the function explicitly; the third and fourth indicate the point by just using a single subscript.

Analogously, fixing $x = x_0$ and letting $y$ vary, we get the partial function $w = f(x_0, y)$, whose graph lies in the vertical plane $x = x_0$, and whose slope at $P$ is the partial derivative of $f$ with respect to $y$; the notations are

\[
\begin{align*}
    \frac{\partial f}{\partial y} \bigg|_{(x_0, y_0)}, & \quad f_y(x_0, y_0), & \quad \frac{\partial w}{\partial y} \bigg|_{(x_0, y_0)}, & \quad \left( \frac{\partial f}{\partial y} \right)_0, & \quad \left( \frac{\partial w}{\partial y} \right)_0.
\end{align*}
\]

The partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ depend on $(x_0, y_0)$ and are therefore functions of $x$ and $y$.

Written as $\partial w/\partial x$, the partial derivative gives the rate of change of $w$ with respect to $x$ alone, at the point $(x_0, y_0)$: it tells how fast $w$ is increasing as $x$ increases, when $y$ is held constant.

For a function of three or more variables, $w = f(x, y, z, \ldots)$, we cannot draw graphs any more, but the idea behind partial differentiation remains the same: to define the partial derivative with respect to $x$, for instance, hold all the other variables constant and take the ordinary derivative with respect to $x$; the notations are the same as above:

\[
\begin{align*}
    \frac{d}{dx} f(x, y_0, z_0, \ldots) & = f_x(x_0, y_0, z_0, \ldots), & \quad \left( \frac{\partial f}{\partial x} \right)_0, & \quad \left( \frac{\partial w}{\partial x} \right)_0.
\end{align*}
\]