Chain Rule and Total Differentials

1. Find the total differential of \( w = x^3yz + xy + z + 3 \) at \((1,2,3)\).

**Answer:** The total differential at the point \((x_0, y_0, z_0)\) is

\[
dw = w_x(x_0, y_0, z_0)\, dx + w_y(x_0, y_0, z_0)\, dy + w_z(x_0, y_0, z_0)\, dz.
\]

In our case,

\[
w_x = 3x^2yz + y, \quad w_y = x^3z + x, \quad w_z = x^3y + 1.
\]

Substituting in the point \((1,2,3)\) we get: \(w_x(1,2,3) = 20\), \(w_y(1,2,3) = 4\), \(w_z(1,2,3) = 3\).

Thus,

\[
dw = 20\, dx + 4\, dy + 3\, dz.
\]

2. Suppose \( w = x^3yz + xy + z + 3 \) and

\[
x = 3\cos t, \quad y = 3\sin t, \quad z = 2t.
\]

Compute \( \frac{dw}{dt} \) and evaluate it at \( t = \pi/2 \).

**Answer:** We do not substitute for \( x, y, z \) before differentiating, so we can practice the chain rule.

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

\[
= (3x^2yz + y)(-3\sin t) + (x^3z + x)(3\cos t) + (x^3y + 1)(2).
\]

At \( t = \pi/2 \) we have \( x = 0, \ y = 3, \ z = \pi, \ \sin \pi/2 = 1, \ \cos \pi/2 = 0 \).

Thus,

\[
\left. \frac{dw}{dt} \right|_{\pi/2} = 3(-3) + 3(0) + (1)2 = -7.
\]

3. Show how the tangent approximation formula leads to the chain rule that was used in the previous problem.

**Answer:** The approximation formula is

\[
\Delta w \approx \left. \frac{\partial f}{\partial x} \right|_o \Delta x + \left. \frac{\partial f}{\partial y} \right|_o \Delta y + \left. \frac{\partial f}{\partial z} \right|_o \Delta z.
\]

If \( x, y, z \) are functions of time then dividing the approximation formula by \( \Delta t \) gives

\[
\frac{\Delta w}{\Delta t} \approx \left. \frac{\partial f}{\partial x} \right|_o \frac{\Delta x}{\Delta t} + \left. \frac{\partial f}{\partial y} \right|_o \frac{\Delta y}{\Delta t} + \left. \frac{\partial f}{\partial z} \right|_o \frac{\Delta z}{\Delta t}.
\]

In the limit as \( \Delta t \to 0 \) we get the chain rule.

Note: we use the regular ‘d’ for the derivative \( \frac{dw}{dt} \) because in the chain of computations

\[
t \to x, y, z \to w
\]

the dependent variable \( w \) is ultimately a function of exactly one independent variable \( t \).

Thus, the derivative with respect to \( t \) is not a partial derivative.