Problems: Lagrange Multipliers

1. Find the maximum and minimum values of \( f(x, y) = x^2 + x + 2y^2 \) on the unit circle.

\[ \text{Answer:} \] The objective function is \( f(x, y) \). The constraint is \( g(x, y) = x^2 + y^2 = 1 \).

Lagrange equations:
\[
\begin{align*}
  f_x &= \lambda g_x \\
  f_y &= \lambda g_y \\
\end{align*}
\]

Constraint:
\[ x^2 + y^2 = 1 \]

The second equation shows \( y = 0 \) or \( \lambda = 2 \).

\( \lambda = 2 \) \Rightarrow \( x = 1/2, y = \pm \sqrt{3}/2 \).

\( y = 0 \) \Rightarrow \( x = \pm 1 \).

Thus, the critical points are \((1/2, \sqrt{3}/2), (1/2, -\sqrt{3}/2), (1, 0)\), and \((-1, 0)\).

\( f(1/2, \pm \sqrt{3}/2) = 9/4 \) (maximum).

\( f(1, 0) = 2 \) (neither min. nor max).

\( f(-1, 0) = 0 \) (minimum).

2. Find the minimum and maximum values of \( f(x, y) = x^2 - xy + y^2 \) on the quarter circle \( x^2 + y^2 = 1, \ x, y \geq 0 \).

\[ \text{Answer:} \] The constraint function here is \( g(x, y) = x^2 + y^2 = 1 \). The maximum and minimum values of \( f(x, y) \) will occur where \( \nabla f = \lambda \nabla g \) or at endpoints of the quarter circle.

\[ \nabla f = (2x - y, -x + 2y) \quad \text{and} \quad \nabla g = (2x, 2y). \]

Setting \( \nabla f = \lambda \nabla g \), we get \( 2x - y = \lambda \cdot 2x \) and \( -x + 2y = \lambda \cdot 2y \).

Solving for \( \lambda \) and setting the results equal to each other gives us:

\[
\begin{align*}
  \frac{2x - y}{2x} &= \frac{-x + 2y}{2y} \\
  \frac{2xy - y^2}{x^2} &= \frac{-x^2 + 2xy}{y^2} \\
\end{align*}
\]

Because we’re constrained to \( x^2 + y^2 = 1 \) with \( x \) and \( y \) non-negative, we conclude that \( x = y = \frac{1}{\sqrt{2}} \).

Thus, the extreme points of \( f(x, y) \) will be at \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (1, 0), \text{or} (0, 1)\).

\[ f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2} \] is the minimum value of \( f \) on this quarter circle.

\[ f(1, 0) = f(0, 1) = 1 \] are the maximal values of \( f \) on this quarter circle.