18.02 Exam 3 – Solutions

1. a) \[
y = 2x, \quad x = 1
\]

\[
y = x
\]

b) \[
\int_0^1 \int_{y/2}^y dx \, dy + \int_1^2 \int_{y/2}^1 dx \, dy.
\]

(the first integral corresponds to the bottom half \(0 \leq y \leq 1\), the second integral to the top half \(1 \leq y \leq 2\).)

2. a) \[
\delta dA = \frac{r \sin \theta}{r^2} r \, dr \, d\theta = \sin \theta \, dr \, d\theta.
\]

\[
M = \int_R \int_R \delta dA = \int_0^\pi \int_0^1 \sin \theta \, dr \, d\theta = \int_0^\pi 2 \sin \theta \, d\theta = \left[-2 \cos \theta\right]_0^\pi = 4.
\]

b) \[
\bar{x} = \frac{1}{M} \int_R \int_R x \delta dA = \frac{1}{4} \int_0^\pi \int_0^1 r \cos \theta \, \sin \theta \, dr \, d\theta
\]

The reason why one knows that \(\bar{x} = 0\) without computation is that the region and the density are symmetric with respect to the y-axis (\(\delta(x, y) = \delta(-x, y)\)).

3. a) \(N_x = -12y = M_y\), hence \(\mathbf{F}\) is conservative.

b) \(f_x = 3x^2 - 6y^2 \Rightarrow f = x^3 - 6y^2 x + c(y) \Rightarrow f_y = -12xy + c'(y) = -12xy + 4y\). So \(c'(y) = 4y\), thus \(c(y) = 2y^2 \) (+ constant). In conclusion

\[
f = x^3 - 6xy^2 + 2y^2 \quad (+ \text{ constant}).
\]

c) The curve \(C\) starts at \((1, 0)\) and ends at \((1, 1)\), therefore

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(1, 0) = (1 - 6 + 2) - 1 = -4.
\]

4. a) The parametrization of the circle \(C\) is \(x = \cos t, \ y = \sin t\), for \(0 \leq t < 2\pi\); then \(dx = -\sin t \, dt, \ dy = \cos t \, dt\) and

\[
W = \int_0^{2\pi} (5 \cos t + 3 \sin t)(-\sin t) \, dt + (1 + \cos(\sin t)) \cos t \, dt.
\]

b) Let \(R\) be the unit disc inside \(C\):

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_R (N_x - M_y) \, dA = \int_R (0 - 3) \, dA = -3 \text{ area}(R) = -3\pi.
\]

5. a) \((0, 4) \quad \begin{array}{c}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{array}
\]

\[
\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_R \text{div} \mathbf{F} \, dxdy
\]

\[
= \int_R (y + \cos x \cos y - \cos x \cos y) \, dxdy = \int_R y \, dxdy
\]

\[
= \int_0^4 \int_0^1 y \, dxdy = \int_0^4 ydy = \left[y^2/2\right]_0^4 = 8.
\]
b) On \( C_4 \), \( x = 0 \), so \( \mathbf{F} = -\sin y \mathbf{j} \), whereas \( \mathbf{n} = -\mathbf{i} \). Hence \( \mathbf{F} \cdot \mathbf{n} = 0 \). Therefore the flux of \( \mathbf{F} \) through \( C_4 \) equals 0. Thus

\[
\int_{C_1+C_2+C_3} \mathbf{F} \cdot \mathbf{n} \, ds = \int_C \mathbf{F} \cdot \mathbf{n} \, ds - \int_{C_4} \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds ;
\]

and the total flux through \( C_1 + C_2 + C_3 \) is equal to the flux through \( C \).

6. Let \( u = 2x - y \) and \( v = x + y - 1 \). The Jacobian

\[
\begin{vmatrix}
 u_x & u_y \\
 v_x & v_y \\
\end{vmatrix}
= \begin{vmatrix}
 2 & -1 \\
 1 & 1 \\
\end{vmatrix}
= 3.
\]

Hence \( dudv = 3dxdy \) and \( dxdy = \frac{1}{3} dudv \), so that

\[
V = \iiint_{(2x-y)^2+(x+y-1)^2<4} (4 - (2x - y)^2 - (x + y - 1)^2) \, dxdy
\]

\[
= \iiint_{u^2+v^2<4} (4 - u^2 - v^2) \frac{1}{3} dudv
\]

\[
= \int_0^{2\pi} \int_0^2 (4 - r^2) \frac{1}{3} r \, rd\theta = \int_0^{2\pi} \left[ \frac{2}{3} r^2 - \frac{1}{12} r^4 \right]_0^2 d\theta
\]

\[
= \int_0^{2\pi} \frac{4}{3} d\theta = \frac{8\pi}{3}.
\]