18.02 Problem Set 7, Part II Solutions

1. (a)

(b)

\[
V = \int_0^4 \int_0^{4-x} \sqrt{4-x} \, dy \, dx \\
= \int_0^4 \left[ y\sqrt{4-x} \right]_{y=0}^{y=4-x} \, dx \\
= \int_0^4 (4-x)^{3/2} \, dx = -\frac{2}{5}(4-x)^{5/2} \bigg|_0^4 = \frac{2}{5}4^{5/2} = \frac{64}{5}.
\]

2. (a) For simplicity let us assume we are integrating the volume of revolution out to some radius \( a \). We also assume that \( f(r) \geq 0 \) for \( 0 \leq r \leq a \). Then if \( R \) is the disc \( x^2 + y^2 \leq a \), we want

\[
V = \int \int_R f \, dA.
\]

In polar coordinates this is

\[
V = \int_0^{2\pi} \int_0^a f(r) r \, dr \, d\theta.
\]

We may write the integral in the other order as well, because the limits to each integral are constants.

\[
V = \int_0^a \left( \int_0^{2\pi} f(r) \, d\theta \right) r \, dr.
\]
Evaluating the inner integral gives
\[ \int_0^a 2\pi rf(r)dr \]
which is the shell method formula.

(b)

3. (a) For a circular sector \( S_\theta \) with center angle \( 2\theta \) and radius \( a \),
\[ A(S_\theta) = \frac{1}{2} a^2 (2\theta) = a^2 \theta \]
and its centroid is at \((\bar{x}_S(\theta), 0)\) where
\[ \bar{x}_S(\theta) = \bar{x}(S_\theta) = \frac{1}{a^2 \theta} \int_{-\theta}^{\theta} \int_0^a (r \cos \varphi) r \, dr \, d\varphi. \]
This comes out to
\[ \bar{x}_S(\theta) = \left( \frac{2 \sin \theta}{3 \theta} \right) a. \]
So we observe a factor \( f_s(\theta) = \frac{2 \sin \theta}{3 \theta} \) governing at what multiple of the radius the centroid must occur.
(b) The result from elementary geometry is that the centroid of a triangular region with uniform density is located at the intersection of the three side-bisectors or ‘medians’, and that this point divides the medians in a ratio of 2 to 1, with the shorter segment nearest the bisection point. Thus we get that for the triangle given and positioned in the same way as the circular sector on the x-axis

\[ \bar{x}_\Delta = \left( \frac{2}{3} \right) a. \]

So \( f_\Delta = \) the factor which multiplies \( a \) is equal to \( \frac{2}{3} \), independent of \( \theta \).

(c) The circular sector region is a subset of the triangular region, with the excess part of the triangle farther away from the origin. Thus we should have \( \bar{x}_S(\theta) < \bar{x}_\Delta \). But in fact the math agrees, since \( \frac{\sin \theta}{\theta} < 1 \), and so the \( f_s(\theta) \), the factor of \( a \) for the sector, which we found in part(a) to be \( f_s(\theta) = \frac{2 \sin \theta}{\theta} \)

thus satisfies the inequality \( f_s(\theta) < \frac{2}{3} = f_\Delta \).

4. Case A: \( (X(x, y, t), Y(x, y, t)) = ((1 + t)x, (1 + t)y) \).

\[ J(x, y, t) = \frac{\partial (X, Y)}{\partial (x, y)} = \begin{bmatrix} 1 + t & 0 \\ 0 & 1 + t \end{bmatrix} \]

and so (a) \( |J(x, y, t)| = (1 + t)^2 \) and (b) \( A(\mathcal{R}_t) = (1 + t)^2 A(\mathcal{R}) \)

Case B: \( (X(x, y, t), Y(x, y, t)) = (x \cos t - y \sin t, x \sin t + y \cos t) \),

\[ J(x, y, t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \]

and so (a) \( |J(x, y, t)| = 1 \) and (b) \( A(\mathcal{R}_t) = A(\mathcal{R}) \) for all \( t \).

Case C: \( (X(x, y, t), Y(x, y, t)) = ((1 + t)x, \left(\frac{1}{1 + t}\right)y) \),

\[ J(x, y, t) = \begin{bmatrix} 1 + t & 0 \\ 0 & \frac{1}{1 + t} \end{bmatrix} \]

and so (a) \( |J(x, y, t)| = 1 \) and (b) \( A(\mathcal{R}_t) = A(\mathcal{R}) \) for all \( t \).

5. Case A: \( (X(x, y, t), Y(x, y, t)) = ((1 + t)x, (1 + t)y) \).

\( \mathbf{v}(x, y, t) = \langle \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} \rangle = \langle x, y \rangle \). The flow lines are straight lines fanning out from the origin. The velocity vectors depend only on the position, and their magnitude increases with the distance from the origin; thus the flow gets faster as it moves away from \( O \).

\( \mathcal{R}_2 \), the points downstream at \( t = 2 \) from the triangular region \( \mathcal{R} \), form a triangular region with vertices at \((0, 0)\), \((3, 3)\) and \((3, -3)\). Thus \( A(\mathcal{R}_2) = \frac{1}{2} \cdot 3 \cdot 6 = 9 = (1 + 2)^2 A(\mathcal{R}) \) as predicted in problem 5, since \( A(\mathcal{R}) = \frac{1}{2} \cdot 1 \cdot 2 \)
1. The flow is not v-i.

Case B: \((X(x, y, t), Y(x, y, t)) = (x \cos t - y \sin t, x \sin t + y \cos t)\). 
\(v(x, y, t) = (-x \sin t - y \cos t, x \cos t - y \sin t)\). The flow lines are circular paths centered at the origin. The velocity vectors depend on position and time; however the speed \(|v(x, y, t)| = \sqrt{x^2 + y^2}\) does not depend explicitly on time; its magnitude increases with the distance from the origin, but the angular velocity \(\omega = \frac{|v|}{r} = 1\) is constant. So the flow is a ‘pure rotating’ circular flow moving counter-clockwise around the origin at 1 rad./unit time.

\(\mathcal{R}_{\pi}\), the points downstream at \(t = \frac{\pi}{2}\) from the triangular region \(\mathcal{R}\), form a triangular region with vertices at \((0, 0), (0, 2)\) and \((-1, 2)\), i.e. the triangular region \(\mathcal{R}\) rotated by \(\frac{\pi}{2}\) counter-clockwise. Thus \(A(\mathcal{R}_{\pi}) = A(\mathcal{R}) = 1\), as predicted in problem 4, since in general the flow is v-i.

Case C: \((X(x, y, t), Y(x, y, t)) = ((1 + t)x, \frac{1}{1+t}y)\). 
\(v(x, y, t) = \langle x, \frac{-y}{(1+t)^2} \rangle\). The flow lines are the hyperbolas \(XY = xy = \text{constant}\), with the x and y-axes as asymptotes. The velocity vectors depend on position and time. (The j-component of the velocity goes to zero as \(t > 0\) increases, which is consistent with the fact that the x-axis is a horizontal asymptote.) The flow comes ‘screaming in’ at high speed from \((0, \infty)\) for \(t > -1\), and then slows down as \(t\) increases.

\(\mathcal{R}_3\), the points downstream at \(t = 3\) from the rectangular region \(\mathcal{R}\), form a rectangular region with vertices at \((4, \frac{1}{4}), (4, 1), (8, \frac{1}{4})\), and \((8, 1)\). Thus \(A(\mathcal{R}_3) = A(\mathcal{R}) = 3\), as predicted in problem 5, since in general the flow is v-i. However, it is not as easy to see why this is the case as it was in case B, where the flow just rotates a region into a congruent region.