Limits in Iterated Integrals

For most students, the trickiest part of evaluating multiple integrals by iteration is to
put in the limits of integration. Fortunately, a fairly uniform procedure is available which
works in any coordinate system. You must always begin by sketching the region; in what
follows we’ll assume you’ve done this.

1. Double integrals in rectangular coordinates.

Let’s illustrate this procedure on the first case that’s usually
taken up: double integrals in rectangular coordinates. Suppose we
want to evaluate over the region $R$ pictured the integral

$$
\int \int_R f(x, y) \, dy \, dx,
$$

we are integrating first with respect to $y$. Then to put in the limits,

1. Hold $x$ fixed, and let $y$ increase (since we are integrating with respect to $y$).
   As the point $(x, y)$ moves, it traces out a vertical line.

2. Integrate from the $y$-value where this vertical line enters the region $R$, to
   the $y$-value where it leaves $R$.

3. Then let $x$ increase, integrating from the lowest $x$-value for which the vertical
   line intersects $R$, to the highest such $x$-value.

Carrying out this program for the region $R$ pictured, the vertical line enters $R$ where
$y = 1 - x$, and leaves where $y = \sqrt{1 - x^2}$.

The vertical lines which intersect $R$ are those between $x = 0$ and
$x = 1$. Thus we get for the limits:

$$
\int \int_R f(x, y) \, dy \, dx = \int_0^1 \int_{\sqrt{1-x^2}}^{1-x} f(x, y) \, dy \, dx.
$$

To calculate the double integral, integrating in the reverse order $\int \int_R f(x, y) \, dx \, dy$,

1. Hold $y$ fixed, let $x$ increase (since we are integrating first with respect to $x$).
   This traces out a horizontal line.

2. Integrate from the $x$-value where the horizontal line enters $R$ to the $x$-value
   where it leaves.

3. Choose the $y$-limits to include all of the horizontal lines which intersect $R$.

Following this prescription with our integral we get:

$$
\int \int_R f(x, y) \, dx \, dy = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy.
$$