Problems: Regions of Integration

1. Find the mass of the region \( R \) bounded by
\[
y = x + 1; \quad y = x^2; \quad x = 0 \text{ and } x = 1, \text{ if density } = \delta(x, y) = xy.
\]

**Answer:**

Inner limits: \( y \) from \( x \) to \( x + 1 \). Outer limits: \( x \) from 0 to 1.

\[
M = \int \int_R \delta(x, y) \, dA = \int_{x=0}^{1} \int_{y=x^2}^{x+1} xy \, dy \, dx
\]

Inner:
\[
\int_{x}^{x+1} xy \, dy = \left[ \frac{y^2}{2} \right]_{x^2}^{x+1} = \frac{x(x + 1)^2}{2} - \frac{x^5}{2} = \frac{2}{2} + \frac{x^3}{2} + x - \frac{x^5}{2}.
\]

Outer:
\[
\int_{0}^{1} \int_{y=x^2}^{x+1} xy \, dy \, dx = \int_{0}^{1} \left[ \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^6}{12} \right]_0^1 = \frac{1}{8} + \frac{1}{4} + \frac{1}{12} = \frac{5}{8}.
\]

**Note:** The syntax \( y = x^2 \) in limits is redundant but useful. We know it must be \( y \) because of the \( dy \) matching the integral sign.

2. Find the volume of the tetrahedron shown below.

**Answer:** The surface has height: \( z = 1 - x - y \).

Limits: inner: \( 0 < y < 1 - x \), outer: \( 0 < x < 1 \). \( V = \int_{x=0}^{1} \int_{y=0}^{1-x} 1 - x - y \, dy \, dx. \)

Inner:
\[
\int_{y=0}^{1-x} 1 - x - y \, dy = y - xy - \frac{y^2}{2} \bigg|_{0}^{1-x} = 1 - x - x^2 - \frac{1}{2} x + \frac{x^2}{2}.
\]

Outer:
\[
\int_{0}^{1} \frac{1}{2} - x + \frac{x^2}{2} \, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.
\]