Moment of Inertia

1. Let $R$ be the triangle with vertices $(0,0)$, $(1,0)$, $(1,\sqrt{3})$ and density $\delta = 1$. Find the polar moment of inertia.

**Answer:** The region $R$ is a 30, 60, 90 triangle.

![Triangle Diagram]

The polar moment of inertia is the moment of inertia around the origin (that is, the z-axis). The figure shows the triangle and a small square piece within $R$. If the piece has area $dA$ then its polar moment of inertia is $dI = r^2 \delta dA$. Summing the contributions of all such pieces and using $\delta = 1$, $dA = r \, dr \, d\theta$, we get the total moment of inertia is

$$I = \iiint_{R} r^2 \delta \, dA = \iiint_{R} r^2 \, r \, dr \, d\theta = \iiint_{R} r^3 \, dr \, d\theta.$$ 

Next we find the limits of integration in polar coordinates. The line

$$x = 1 \iff r \cos \theta = 1 \iff r = \sec \theta.$$ 

So, using radial stripes, the limits are: (inner) $r$ from 0 to $\sec \theta$; (outer) $\theta$ from 0 to $\pi/3$.

Thus,

$$I = \int_{0}^{\pi/3} \int_{0}^{\sec \theta} r^3 \, dr \, d\theta.$$ 

Inner integral: \[\frac{\sec^4 \theta}{4}\].

Outer integral: Use $\sec^4 \theta = \sec^2 \theta \sec^2 \theta = (1 + \tan^2 \theta) \, d(\tan \theta) \Rightarrow$ the outer integral is

$$\frac{1}{4} \left( \tan \theta + \frac{\tan^3 \theta}{3} \right) \bigg|_{0}^{\pi/3} = \frac{1}{4} \left( \sqrt{3} + \frac{(\sqrt{3})^3}{3} \right) = \frac{\sqrt{3}}{2}.$$ 

The polar moment of inertia is $\frac{\sqrt{3}}{2}$. 

