Changing Variables in Multiple Integrals

1. Changing variables.

Double integrals in $x, y$ coordinates which are taken over circular regions, or have integrands involving the combination $x^2 + y^2$, are often better done in polar coordinates:

$$\int \int_R f(x, y) dA = \int \int_R g(r, \theta) r \,dr \,d\theta .$$

This involves introducing the new variables $r$ and $\theta$, together with the equations relating them to $x, y$ in both the forward and backward directions:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x); \quad x = r \cos \theta, \quad y = r \sin \theta .$$

Changing the integral to polar coordinates then requires three steps:

**A.** Changing the integrand $f(x, y)$ to $g(r, \theta)$, by using (2);

**B.** Supplying the area element in the $r, \theta$ system: $dA = r \,dr \,d\theta$;

**C.** Using the region $R$ to determine the limits of integration in the $r, \theta$ system.

In the same way, double integrals involving other types of regions or integrands can sometimes be simplified by changing the coordinate system from $x, y$ to one better adapted to the region or integrand. Let’s call the new coordinates $u$ and $v$; then there will be equations introducing the new coordinates, going in both directions:

$$u = u(x, y), \quad v = v(x, y); \quad x = x(u, v), \quad y = y(u, v)$$

(often one will only get or use the equations in one of these directions). To change the integral to $u, v$-coordinates, we then have to carry out the three steps A, B, C above. A first step is to picture the new coordinate system; for this we use the same idea as for polar coordinates, namely, we consider the grid formed by the level curves of the new coordinate functions:

$$u(x, y) = u_0, \quad v(x, y) = v_0 .$$

Once we have this, algebraic and geometric intuition will usually handle steps A and C, but for B we will need a formula: it uses a determinant called the **Jacobian**, whose notation and definition are

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}. $$

Using it, the formula for the area element in the $u, v$-system is

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \,dv ,$$
so the change of variable formula is

$\int \int_R f(x, y) \, dx \, dy = \int \int_R g(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv$,

where $g(u, v)$ is obtained from $f(x, y)$ by substitution, using the equations (3).

We will derive the formula (5) for the new area element in the next section; for now let’s check that it works for polar coordinates.

**Example 1.** Verify (1) using the general formulas (5) and (6).

**Solution.** Using (2), we calculate:

$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r (\cos^2 \theta + \sin^2 \theta) = r$,

so that $dA = r \, dr \, d\theta$, according to (5) and (6); note that we can omit the absolute value, since by convention, in integration problems we always assume $r \geq 0$, as is implied already by the equations (2).

We now work an example illustrating why the general formula is needed and how it is used; it illustrates step C also — putting in the new limits of integration.

**Example 2.** Evaluate $\int \int_R \left( \frac{x - y}{x + y + 2} \right)^2 \, dx \, dy$ over the region $R$ pictured.

**Solution.** This would be a painful integral to work out in rectangular coordinates. But the region is bounded by the lines

$x + y = \pm 1, \quad x - y = \pm 1$

and the integrand also contains the combinations $x - y$ and $x + y$. These powerfully suggest that the integral will be simplified by the change of variable (we give it also in the inverse direction, by solving the first pair of equations for $x$ and $y$):

$u = x + y, \quad v = x - y; \quad x = \frac{u + v}{2}, \quad y = \frac{u - v}{2}$.

We will also need the new area element; using (5) and (9) above, we get

$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$;

note that it is the second pair of equations in (9) that were used, not the ones introducing $u$ and $v$. Thus the new area element is (this time we do need the absolute value sign in (6))

$dA = \frac{1}{2} \, du \, dv$.

We now combine steps A and B to get the new double integral; substituting into the integrand by using the first pair of equations in (9), we get

$\int \int_R \left( \frac{x - y}{x + y + 2} \right)^2 \, dx \, dy = \int \int_R \left( \frac{v}{u + 2} \right)^2 \frac{1}{2} \, du \, dv$.
In $uv$-coordinates, the boundaries (8) of the region are simply $u = \pm 1$, $v = \pm 1$, so the integral (12) becomes

$$\int \int_{R} \left( \frac{v}{u+2} \right)^2 \frac{1}{2} \, du \, dv = \int_{-1}^{1} \int_{-1}^{1} \left( \frac{v}{u+2} \right)^2 \frac{1}{2} \, du \, dv$$

We have

inner integral $= -\frac{v^2}{2(u+2)} \bigg|_{u=-1}^{u=1} = \frac{v^2}{3}$; 
outer integral $= \frac{v^3}{9} \bigg|_{-1}^{1} = \frac{2}{9}$. 
