4. Line Integrals in the Plane

4A. Plane Vector Fields

4A-1

a) All vectors in the field are identical; continuously differentiable everywhere.
b) The vector at $P$ has its tail at $P$ and head at the origin; field is cont. diff. everywhere.
c) All vectors have unit length and point radially outwards; cont. diff. except at $(0,0)$.
d) Vector at $P$ has unit length, and the clockwise direction perpendicular to $OP$.

4A-2

a) $a \mathbf{i} + b \mathbf{j}$  
b) $\frac{x \mathbf{i} + y \mathbf{j}}{r^2}$  
c) $f'(x) \frac{x \mathbf{i} + y \mathbf{j}}{r}$

4A-3

a) $i + 2j$  
b) $-r(x \mathbf{i} + y \mathbf{j})$  
c) $\frac{y \mathbf{i} - x \mathbf{j}}{r^3}$  
d) $f(x, y)(i + j)$

4A-4

$k \cdot \frac{-y \mathbf{i} + x \mathbf{j}}{r^2}$

4B. Line Integrals in the Plane

4B-1

a) On $C_1$: $y = 0$, $dy = 0$; therefore
$$\int_{C_1} \left(x^2 - y^2\right) dx + 2x \ dy = \int_{-1}^{1} x^2 \ dx = \frac{x^3}{3} \big|_{-1}^{1} = \frac{2}{3}.$$

On $C_2$: $y = 1 - x^2$, $dy = -2x \ dx$;
$$\int_{C_2} \left(x^2 - y^2\right) dx + 2x \ dy = \int_{-1}^{1} (2x^2 - 1) \ dx - 4x^2 \ dx$$
$$= \int_{-1}^{1} (-2x^2 - 1) \ dx = -\left[\frac{2}{3}x^3 + x\right]_{-1}^{1} = -\frac{4}{3} - 2 = -\frac{10}{3}.$$

b) $C$: use the parametrization $x = \cos t$, $y = \sin t$; then $dx = -\sin t \ dt$, $dy = \cos t \ dt$
$$\int_{C} xy \ dx - x^2 \ dy = \int_{\pi/2}^{0} -\sin^2 t \cos t \ dt - \cos^2 t \cos t \ dt = -\int_{\pi/2}^{0} \cos t \ dt = -\sin t \big|_{\pi/2}^{0} = 1.$$

c) $C = C_1 + C_2 + C_3$: $C_1 : x = dx = 0$; $C_2 : y = 1 - x$; $C_3 : y = dy = 0$
$$\int_{C} y \ dx - x \ dy = \int_{C_1} 0 + \int_{C_2} (1-x) dx - x(-dx) + \int_{C_3} 0 = \int_{0}^{1} dx = 1.$$  
d) $C : x = 2 \cos t$, $y = \sin t$; $dx = -2 \sin t \ dt$  
$$\int_{C} y \ dx = \int_{0}^{2\pi} -2 \sin^2 t \ dt = -2\pi.$$  
e) $C : x = t^2$, $y = t^3$; $dx = 2t \ dt$, $dy = 3t^2 \ dt$
$$\int_{C} 6y \ dx + x \ dy = \int_{1}^{2} 6t^3 (2t \ dt) + t^2 (3t^2 \ dt) = \int_{1}^{2} (15t^4) \ dt = 3t^5 \big|_{1}^{2} = 3 \cdot 31.$$  
f) $\int_{C} (x + y) \ dx + xy \ dy = \int_{C_1} 0 + \int_{0}^{1} (x + 2) \ dx = \frac{x^2}{2} + 2x \big|_{0}^{1} = \frac{5}{2}.$

4B-2

a) The field $\mathbf{F}$ points radially outward, the unit tangent $\mathbf{t}$ to the circle is always perpendicular to the radius; therefore $\mathbf{F} \cdot \mathbf{t} = 0$ and $\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot \mathbf{t} \ ds = 0$

b) The field $\mathbf{F}$ is always tangent to the circle of radius $a$, in the clockwise direction, and of magnitude $a$. Therefore $\mathbf{F} = -a \mathbf{t}$, so that $\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot \mathbf{t} \ ds = -\int_{C} a \ ds = -2\pi a^2$. 
4B-3  a) maximum if $C$ is in the direction of the field: $C = \frac{i + j}{\sqrt{2}}$

b) minimum if $C$ is in the opposite direction to the field: $C = -\frac{i + j}{\sqrt{2}}$

c) zero if $C$ is perpendicular to the field: $C = \pm \frac{i - j}{\sqrt{2}}$

d) max = $\sqrt{2}$, min = $-\sqrt{2}$: by (a) and (b), for the max or min $\mathbf{F}$ and $C$ have respectively the same or opposite constant direction, so $\int_C \mathbf{F} \cdot d\mathbf{r} = \pm |\mathbf{F}| \cdot |C| = \pm \sqrt{2}$.

4C. Gradient Fields and Exact Differentials

4C-1 a) $\mathbf{F} = \nabla f = 3x^2y \mathbf{i} + (x^3 + y^2) \mathbf{j}$

b) (i) Using $y$ as parameter, $C_1$ is: $x = y^2$, $y = y$; thus $dx = 2y dy$, and

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} 3(y^2)^2y \cdot 2y dy + [(y^2)^2 + 3y^2] dy = \int_{-1}^{1} (7y^6 + 3y^2) dy = (y^7 + y^3)|_{-1}^{1} = 4.$$  

b) (ii) Using $y$ as parameter, $C_2$ is: $x = 1$, $y = y$; thus $dx = 0$, and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} (1 + 3y^2) dy = (y + y^3)|_{-1}^{1} = 4.$$  

b) (iii) By the Fundamental Theorem of Calculus for line integrals,  

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$  

Here $A = (1, -1)$ and $B = (1, 1)$, so that $\int_C \nabla f \cdot d\mathbf{r} = (1 + 1) - (-1 - 1) = 4$.

4C-2 a) $\mathbf{F} = \nabla f = (xye^{xy} + e^{xy}) \mathbf{i} + (x^2e^{xy}) \mathbf{j}$.

b) (i) Using $x$ as parameter, $C$ is: $x = x$, $y = 1/x$, so $dy = -dx/x^2$, and so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{1}^{0} \left(e + e\right) dx + (x^2e)(-dx/x^2) = (2ex - ex)|_{1}^{0} = -e.$$  

b) (ii) Using the F.T.C. for line integrals, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, \infty) = 0 - e = -e$.

4C-3 a) $\mathbf{F} = \nabla f = (\cos x \cos y) \mathbf{i} - (\sin x \sin y) \mathbf{j}$.

b) Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent, for any $C$ connecting $A : (x_0, y_0)$ to $B : (x_1, y_1)$, we have by the F.T.C. for line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \sin x_1 \cos y_1 - \sin x_0 \cos y_0$$  

This difference on the right-hand side is maximized if $\sin x_1 \cos y_1$ is maximized, and $\sin x_0 \cos y_0$ is minimized. Since $|\sin x \cos y| = |\sin x||\cos y| \leq 1$, the difference on the right hand side has a maximum of 2, attained when $\sin x_1 \cos y_1 = 1$ and $\sin x_0 \cos y_0 = -1$.

(For example, a $C$ running from $(-\pi/2, 0)$ to $(\pi/2, 0)$ gives this maximum value.)
4C-5  a) \( \mathbf{F} \) is a gradient field only if \( M_y = N_x \), that is, if \( 2y = ay \), so \( a = 2 \).
By inspection, the potential function is \( f(x, y) = xy^2 + x^2 + c \); you can check that \( \mathbf{F} = \nabla f \).

b) The equation \( M_y = N_x \) becomes \( e^{x+y}(x+a) = x e^{x+y} + e^{x+y} \), which is \( e^{x+y}(x+1) \).
Therefore \( a = 1 \).

To find the potential function \( f(x, y) \), using Method 2 we have
\[
f_x = e^y e^x (x + 1) \Rightarrow f(x, y) = e^y x e^x + g(y).
\]
Differentiating, and comparing the result with \( N \), we find
\[
f_y = e^y x e^x + g'(y) = x e^{x+y}; \text{ therefore } g'(y) = 0, \text{ so } g(y) = c \text{ and } f(x, y) = x e^{x+y} + c.
\]

4C-6  a) \( ydx - xdy \) is not exact, since \( M_y = 1 \) but \( N_x = -1 \).

b) \( y(2x + y) dx + x(2y + x) dy \) is exact, since \( M_y = 2x + 2y = N_x \).

Using Method 1 to find the potential function \( f(x, y) \), we calculate the line integral over the standard broken line path shown, \( C = C_1 + C_2 \).

\[
f(x_1, y_1) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0)}^{(x_1,y_1)} y(2x + y) dx + x(2y + x) dy.
\]
On \( C_1 \) we have \( y = 0 \) and \( dy = 0 \), so \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0 \).

On \( C_2 \), we have \( x = x_1 \) and \( dx = 0 \), so \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{y_1} x_1(2y + x_1) dx = x_1 y_1^2 + x_1^2 y_1 \).

Therefore, \( f(x, y) = x^2 y + xy^2 \); to get all possible functions, add \(+c\).

4D. Green’s Theorem

4D-1  a) Evaluating the line integral first, we have \( C : \ x = \cos t, \ y = \sin t \), so
\[
\oint_C 2y \, dx + x \, dy = \int_0^{2\pi} (-2 \sin^2 t + \cos^2 t) \, dt = \int_0^{2\pi} (1 - 3 \sin^2 t) \, dt = t - 3 \left( \frac{t}{2} - \frac{\sin 2t}{4} \right) \bigg|_0^{2\pi} = -\pi.
\]

For the double integral over the circular region \( R \) inside the \( C \), we have
\[
\iint_R (N_x - M_y) \, dA = \iint_R (1 - 2) \, dA = \text{area of } R = -\pi.
\]

b) Evaluating the line integral, over the indicated path \( C = C_1 + C_2 + C_3 + C_4 \),
\[
\oint_C x^2 \, dx + x^2 \, dy = \int_0^1 x^2 \, dx + \int_0^1 4 \, dy + \int_2^1 x^2 \, dx + \int_1^0 0 \, dy = 4,
\]
since the first and third integrals cancel, and the fourth is 0.

For the double integral over the rectangle \( R \),
\[
\iint_R 2x \, dA = \int_0^2 \int_0^1 2x \, dy \, dx = x^2 \bigg|_0^2 = 4.
\]
c) Evaluating the line integral over \( C = C_1 + C_2 \), we have
\[
C_1 : \ x = x, \ y = x^2; \quad \int_{C_1} xy \, dx + y^2 \, dy = \int_0^1 x \cdot x^2 \, dx + x^4 \cdot 2x \, dx = \left. \frac{x^4}{4} + \frac{x^6}{3} \right|_0^1 = \frac{7}{12}
\]
\[
C_2 : \ x = x, \ y = x; \quad \int_{C_2} xy \, dx + y^2 \, dy = \int_1^0 (x^2 \, dx + x^2 \, dx) = \left. \frac{2}{3} x^3 \right|_1^0 = -\frac{2}{3}.
\]
Therefore, \( \int_C xy \, dx + y^2 \, dy = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12} \).

Evaluating the double integral over the interior \( R \) of \( C \), we have
\[
\iint_R -x \, dA = \int_0^1 \int_{x^2}^x -x \, dy \, dx;
\]
evaluating: Inner: \( -xy \bigg|_{y=x^2}^{y=x} = -x^2 + x^3 \); Outer: \( -\frac{x^3}{3} + \frac{x^4}{4} \bigg|_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12} \).

4D-2 By Green’s theorem, \( \oint_C 4x^3 y \, dx + x^4 \, dy = \iint_R (4x^3 - 4x^3) \, dA = 0 \).

This is true for every closed curve \( C \) in the plane, since \( M \) and \( N \) have continuous derivatives for all \( x, y \).

4D-3 We use the symmetric form for the integrand since the parametrization of the curve does not favor \( x \) or \( y \); this leads to the easiest calculation.
\[
\text{Area} = \frac{1}{2} \oint_C -y \, dx + x \, dy = \frac{1}{2} \int_0^{2\pi} 3 \sin^4 t \cos^2 t \, dt + 3 \sin^2 t \cos^4 t \, dt = \frac{3}{2} \int_0^{2\pi} \sin^2 t \cos^2 t \, dt
\]
Using \( \sin^2 t \cos^2 t = \frac{1}{4} (\sin 2t)^2 = \frac{1}{4} \cdot \frac{1}{2} (1 - \cos 4t) \), the above equals \( \frac{3}{8} \left( \frac{t}{2} - \frac{\sin 4t}{8} \right)^{2\pi} = \frac{3\pi}{8} \).

4D-4 By Green’s theorem, \( \oint_C -y^3 \, dx + x^3 \, dy = \iint_R (3x^2 + 3y^2) \, dA > 0 \), since the integrand is always positive outside the origin.

4D-5 Let \( C \) be a square, and \( R \) its interior. Using Green’s theorem,
\[
\oint_C xy^2 \, dx + (x^2 y + 2x) \, dy = \iint_R (2xy + 2 - 2xy) \, dA = \iint_R 2 \, dA = 2(\text{area of } R).
\]

4E. Two-dimensional Flux

4E-1 The vector \( \mathbf{F} \) is the velocity vector for a rotating disc; it is at each point tangent to the circle centered at the origin and passing through that point.

a) Since \( \mathbf{F} \) is tangent to the circle, \( \mathbf{F} \cdot \mathbf{n} = 0 \) at every point on the circle, so the flux is 0.

b) \( \mathbf{F} = x \mathbf{j} \) at the point \((x,0)\) on the line. So if \( x_0 > 0 \), the flux at \( x_0 \) has the same magnitude as the flux at \(-x_0\) but the opposite sign, so the net flux over the line is 0.

c) \( \mathbf{n} = -\mathbf{j} \), so \( \mathbf{F} \cdot \mathbf{n} = x \mathbf{j} \cdot -\mathbf{j} = -x \). Thus \( \int \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^1 -x \, dx = -\frac{1}{2} \).
4E-2 All the vectors of $\mathbf{F}$ have length $\sqrt{2}$ and point northeast. So the flux across a line segment $C$ of length 1 will be

a) maximal, if $C$ points northwest;

b) minimal, if $C$ points southeast;

c) zero, if $C$ points northeast or southwest;

d) $-1$, if $C$ has the direction and magnitude of $\mathbf{i}$ or $-\mathbf{j}$; the corresponding normal vectors are then respectively $-\mathbf{j}$ and $-\mathbf{i}$, by convention, so that $\mathbf{F} \cdot \mathbf{n} = (\mathbf{i} + \mathbf{j}) \cdot \mathbf{-j} = -1$. or $(\mathbf{i} + \mathbf{j}) \cdot \mathbf{-i} = -1$.

e) respectively $\sqrt{2}$ and $-\sqrt{2}$, since the angle $\theta$ between $\mathbf{F}$ and $\mathbf{n}$ is respectively 0 and $\pi$, so that respectively $\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}| \cos \theta = \pm \sqrt{2}$.

4E-3 $\int_C M \, dy - N \, dx = \int_C x^2 \, dy - xy \, dx = \int_0^1 (t+1)^2 \,dt - (t+1) \, dt$

\[= \int_0^1 (t^3 + 3t^2 + 2t) \, dt = \left[ \frac{t^4}{4} + t^3 + t^2 \right]_0^1 = \frac{9}{4}. \]

4E-4 Taking the curve $C = C_1 + C_2 + C_3 + C_4$ as shown,

\[\int_C x \, dy - y \, dx = \int_{C_1} 0 + \int_0^1 -dx + \int_1^{0} dy + \int_{C_4} 0 = -2. \]

4E-5 Since $\mathbf{F}$ and $\mathbf{n}$ both point radially outwards, $\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}| = a^m$, at every point of the circle $C$ of radius $a$ centered at the origin.

a) The flux across $C$ is $a^m \cdot 2\pi a = 2\pi a^{m+1}$.

b) The flux will be independent of $a$ if $m = -1$.

4F. Green’s Theorem in Normal Form

4F-1 a) both are 0  b) div $\mathbf{F} = 2x + 2y$; curl $\mathbf{F} = 0$  c) div $\mathbf{F} = x + y$; curl $\mathbf{F} = y - x$

4F-2 a) div $\mathbf{F} = (-\omega y)_x + (\omega x)_y = 0$; curl $\mathbf{F} = (\omega x)_x - (-\omega y)_y = 2\omega$.

b) Since $\mathbf{F}$ is the velocity field of a fluid rotating with constant angular velocity (like a rigid disc), there are no sources or sinks: fluid is not being added to or subtracted from the flow at any point.

c) A paddlewheel placed at the origin will clearly spin with the same angular velocity $\omega$ as the rotating fluid, so by Notes V4.11, the curl should be $2\omega$ at the origin. (It is much less clear that the curl is $2\omega$ at all other points as well.)

4F-3 The line integral for flux is $\int_C x \, dy - y \, dx$; its value is 0 on any segment of the $x$-axis since $y = dy = 0$; on the upper half of the unit semicircle (oriented counterclockwise), $\mathbf{F} \cdot \mathbf{n} = 1$, so the flux is the length of the semicircle: $\pi$.

Letting $R$ be the region inside $C$, $\int_0 \int_R \text{div} \mathbf{F} \, dA = \int_0 \int_R 2 \, dA = 2(\pi/2) = \pi$.

4F-4 For the flux integral $\int_C x^2 \, dy - xy \, dx$ over $C = C_1 + C_2 + C_3 + C_4$, we get for the four sides respectively $\int_{C_1} 0 + \int_0^1 dy + \int_1^0 -x \, dx + \int_{C_4} 0 = \frac{3}{2}$. 

4. LINE INTEGRALS IN THE PLANE
For the double integral, \( \int \int_R \text{div} \mathbf{F} \, dA = \int \int_R 3x \, dA = \int_0^1 \int_0^1 3x \, dy \, dx = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} \).

4F-5 \quad r = (x^2 + y^2)^{1/2} \quad \Rightarrow \quad r_x = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r}; \quad \text{by symmetry, } r_y = \frac{y}{r}.

To calculate \( \text{div} \mathbf{F} \), we have \( M = r^n x \) and \( N = r^n y \); therefore by the chain rule, and the above values for \( r_x \) and \( r_y \), we have

\[
M_x = r^n + nr^{n-1}x \cdot \frac{x}{r} = r^n + nr^{n-2}x^2; \quad \text{similarly (or by symmetry),}
\]

\[
N_y = r^n + nr^{n-1}y \cdot \frac{y}{r} = r^n + nr^{n-2}y^2, \quad \text{so that}
\]

\[
\text{div} \mathbf{F} = M_x + N_y = 2r^n + nr^{n-2}(x^2 + y^2) = r^n(2 + n), \quad \text{which } = 0 \text{ if } n = -2.
\]

To calculate \( \text{curl} \mathbf{F} \), we have by the chain rule

\[
N_x = nr^{n-1} \cdot \frac{x}{r} \cdot y; \quad M_y = nr^{n-1} \cdot \frac{y}{r} \cdot x, \quad \text{so that } \text{curl} \mathbf{F} = N_x - M_y = 0, \text{ for all } n.
\]

4G. Simply-connected Regions

4G-1 \quad \text{Hypotheses: the region } R \text{ is simply connected, } \mathbf{F} = M \mathbf{i} + N \mathbf{j} \text{ has continuous derivatives in } R, \text{ and } \text{curl} \mathbf{F} = 0 \text{ in } R.

\text{Conclusion: } \mathbf{F} \text{ is a gradient field in } R \quad (\text{or, } M \, dx + N \, dy \text{ is an exact differential}).

\begin{enumerate}
\item \( \text{curl} \mathbf{F} = 2y - 2y = 0, \text{ and } R \text{ is the whole } xy\text{-plane. Therefore } \mathbf{F} = \nabla f \text{ in the plane.} \)
\item \( \text{curl} \mathbf{F} = -y \sin x - x \sin y \neq 0, \text{ so the differential is not exact.} \)
\item \( \text{curl} \mathbf{F} = 0, \text{ but } R \text{ is the exterior of the unit circle, which is not simply-connected; criterion fails.} \)
\item \( \text{curl} \mathbf{F} = 0, \text{ and } R \text{ is the interior of the unit circle, which is simply-connected, so the differential is exact.} \)
\item \( \text{curl} \mathbf{F} = 0 \text{ and } R \text{ is the first quadrant, which is simply-connected, so } \mathbf{F} \text{ is a gradient field.} \)
\end{enumerate}

4G-2 \quad \begin{enumerate}
\item \( f(x, y) = xy^2 + 2x \quad \text{b) } f(x, y) = \frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2} \)
\end{enumerate}

\text{c) Using Method 1, we take the origin as the starting point and use the straight line to } (x_1, y_1) \text{ as the path } C. \text{ In polar coordinates, } x_1 = r_1 \cos \theta_1, \quad y_1 = r_1 \sin \theta_1; \text{ we use } r \text{ as the parameter, so the path is } C: x = r \cos \theta, \quad y = r \sin \theta, \quad 0 \leq r \leq r_1. \text{ Then}

\[
f(x_1, y_1) = \int_C \frac{x \, dx + y \, dy}{\sqrt{1 - r^2}} = \int_0^{r_1} \frac{r \cos^2 \theta + r \sin^2 \theta}{\sqrt{1 - r^2}} \, dr
\]

\[
= \int_0^{r_1} \frac{r \, dr}{\sqrt{1 - r^2}} = -\sqrt{1 - r^2}\bigg|_0^{r_1} = -\sqrt{1 - r_1^2} + 1.
\]

\text{Therefore,}

\[
\frac{x \, dx + y \, dy}{\sqrt{1 - r^2}} = d(-\sqrt{1 - r^2}).
\]

\text{Another approach: } x \, dx + y \, dy = \frac{1}{2}d(r^2); \text{ therefore}

\[
\frac{x \, dx + y \, dy}{\sqrt{1 - r^2}} = \frac{1}{2} \cdot \frac{d(r^2)}{\sqrt{1 - r^2}} = d(-\sqrt{1 - r^2}).
\]

(Think of \( r^2 \) as a new variable \( u \), and integrate.)
4G-3 By Example 3 in Notes V5, we know that \( \mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{r^3} = \nabla \left( -\frac{1}{r} \right) \).

Therefore, \( \int_{(1,1)}^{(3,4)} \frac{-1}{r} \sqrt{2} = \frac{1}{\sqrt{2}} - \frac{1}{5} \).

4G-4 By Green’s theorem \( \int_C xy\,dx + x^2\,dy = \int\int_R x\,dA \).

For any plane region of density 1, we have \( \int\int_R x\,dA = \bar{x}\cdot\text{(area of } R) \), where \( \bar{x} \) is the x-component of its center of mass. Since our region is symmetric with respect to the y-axis, its center of mass is on the y-axis, hence \( \bar{x} = 0 \) and so \( \int\int_R x\,dA = 0 \).

4G-5

a) yes
b) no (a circle surrounding the line segment lies in \( R \), but its interior does not)
c) yes (no finite curve could surround the entire positive x-axis)
d) no (the region does not consist of one connected piece)
e) yes if \( \theta_0 < 2\pi \); no if \( \theta_0 \geq 2\pi \), since then \( R \) is the plane with (0, 0) removed
f) no (a circle between the two boundary circles lies in \( R \), but its interior does not)
g) yes

4G-6

a) continuously differentiable for \( x, y > 0 \); thus \( R \) is the first quadrant without the two axes, which is simply-connected.
b) continuous differentiable if \( r < 1 \); thus \( R \) is the interior of the unit circle, and is simply-connected.
c) continuously differentiable if \( r > 1 \); thus \( R \) is the exterior of the unit circle, and is not simply-connected.
d) continuously differentiable if \( r \neq 0 \); thus \( R \) is the plane with the origin removed, and is not simply-connected.
e) continuously differentiable if \( r \neq 0 \); same as (d).

4H. Multiply-connected Regions

4H-1 a) 0; 0 b) 2; 4\pi c) -1; -2\pi d) -2; -4\pi

4H-2 In each case, the winding number about each of the points is given, then the value of the line integral of \( \mathbf{F} \) around the curve.

a) \((1, -1, 1); \quad 2 - \sqrt{2} + \sqrt{3} \)
b) \((-1, 0, 1); \quad -2 + \sqrt{3} \)
c) \((-1, 0, 0); \quad -2 \)
d) \((-1, -2, 1); \quad -2 - 2\sqrt{2} + \sqrt{3} \)