V2.1 Gradient Fields and Exact Differentials

1. Criterion for gradient fields.

Let \( \mathbf{F} = M(x, y) \mathbf{i} + N(x, y) \mathbf{j} \) be a two-dimensional vector field, where \( M \) and \( N \) are continuous functions. There are three equivalent ways of saying that \( \mathbf{F} \) is conservative, i.e., a gradient field:

\[
(1) \quad \mathbf{F} = \nabla f \iff \int_p^Q \mathbf{F} \cdot d\mathbf{r} \text{ is path-independent} \iff \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for any closed } C
\]

Unfortunately, these equivalent formulations don’t give us any effective way of deciding if a given field \( \mathbf{F} \) is a conservative field or not. However, if we assume that \( \mathbf{F} \) is not just continuous but is even continuously differentiable (meaning: \( M_x, M_y, N_x, N_y \) all exist and are continuous), then there is a simple and elegant criterion for deciding whether or not \( \mathbf{F} \) is a gradient field in some region.

**Criterion.** Let \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} \) be continuously differentiable in a region \( D \). Then, in \( D \),

\[
(2) \quad \mathbf{F} = \nabla f \quad \text{for some } f(x, y) \Rightarrow M_y = N_x .
\]

**Proof.** Since \( \mathbf{F} = \nabla f \), this means

\[
M = f_x \quad \text{and} \quad N = f_y .
\]

Therefore,

\[
M_y = f_{xy} \quad \text{and} \quad N_x = f_{yx} .
\]

But since these two mixed partial derivatives are continuous (since \( M_y \) and \( N_x \) are, by hypothesis), a standard 18.02 theorem says they are equal. Thus \( M_y = N_x \). \( \square \)

This theorem may be expressed in a slightly different form, if we define the scalar function called the **two-dimensional curl** of \( \mathbf{F} \) by

\[
(3) \quad \text{curl } \mathbf{F} = N_x - M_y .
\]

Then (2) becomes

\[
(2') \quad \mathbf{F} = \nabla f \Rightarrow \text{curl } \mathbf{F} = 0 .
\]

This criterion allows us to test \( \mathbf{F} \) to see if it is a gradient field. Naturally, we would also like to know that the converse is true: if \( \text{curl } \mathbf{F} = 0 \), then \( \mathbf{F} \) is a gradient field. As we shall see, however, this requires some additional hypotheses on the region \( D \). For now, we will assume \( D \) is the whole plane. Then we have

**Converse to Criterion.** Let \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} \) be continuously differentiable for all \( x, y \).

\[
(4) \quad M_y = N_x \text{ for all } x, y \Rightarrow \mathbf{F} = \nabla f \text{ for some differentiable } f \text{ and all } x, y .
\]

The proof of (4) will be postponed until we have more technique. For now we will illustrate the use of the criterion and its converse.
Example 1. For which value(s), if any of the constants $a, b$ will $axy \mathbf{i} + (x^2 + by) \mathbf{j}$ be a gradient field?

Solution. The partial derivatives are continuous for all $x, y$ and $M_y = ax, \ N_x = 2x$. Thus by (2) and (4), $F = \nabla f \iff a = 2; \ b$ is arbitrary.

Example 2. Are the fields $F = \frac{x \mathbf{i} + y \mathbf{j}}{x^2 + y^2}, \ \ G = \frac{-y \mathbf{i} + x \mathbf{j}}{x^2 + y^2}$ conservative?

Solution. We have (the second line follows from the first by interchanging $x$ and $y$):

$$\frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) = \frac{-x^2 + y^2}{(x^2 + y^2)^2}; \quad \frac{\partial}{\partial x} \left( \frac{y}{x^2 + y^2} \right) = \frac{2xy}{(x^2 + y^2)^2};$$

$$\frac{\partial}{\partial y} \left( \frac{x}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2}; \quad \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) = \frac{2yx}{(x^2 + y^2)^2};$$

from this, we see immediately that

$$\frac{\partial}{\partial y} \left( \frac{x}{x^2 + y^2} \right) = \frac{\partial}{\partial x} \left( \frac{-y}{x^2 + y^2} \right); \quad \frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right);$$

the two equations in the last line show respectively that $F$ and $G$ satisfy the criterion (2). However, neither field is defined at $(0, 0)$, so that the converse (4) is not applicable. So the question cannot be decided just on the basis of (2) and (4). In fact, it turns out that $F$ is a gradient field, since one can check that

$$F = \nabla \ln \sqrt{x^2 + y^2} = \nabla \ln r.$$ 

On the other hand, $G$ is not conservative, since if $C$ is the unit circle $x = \cos t, \ y = \sin t,$ we have

$$\oint_C G \cdot d\mathbf{r} = \oint_C \frac{-y \, dx + x \, dy}{x^2 + y^2} = \int_0^{2\pi} \frac{\sin^2 t \, dt + \cos^2 t \, dt}{\sin^2 t + \cos^2 t} = 2\pi \neq 0.$$

We will return to this example later on.