Identifying Gradient Fields and Exact Differentials

1. Determine whether each of the vector fields below is conservative.
   a) $F = (xe^x + y, x)$
   b) $F = (xe^x + y, x + 2)$
   c) $F = (xe^x + y + x, x)$

   **Answer:** We know from lecture that if $F = Mi + Nj$ is continuously differentiable for all $x$ and $y$, then $M_y = N_x$ for all $x$ and $y$ $\implies$ $F$ is conservative.

   Each of the fields in question is continuously differentiable for all $x$ and $y$.
   a) $M = xe^x + y$, $N = x$. $M_y = 1$, $N_x = 1$. The field is conservative.
   b) $M = xe^x + y$, $N = x + 2$. $M_y = 1$, $N_x = 1$. The field is conservative.
   c) $M = xe^x + y + x$, $N = x$. $M_y = 1$, $N_x = 1$. The field is conservative.

   In fact, we can add any function of $x$ to $M$ and any function of $y$ to $N$ without affecting $M_y$ and $N_x$.

2. Show $(xe^x + y) \, dx + x \, dy$ is exact.

   **Answer:** We know from lecture that $M \, dx + N \, dy$ is an exact differential if and only if $F = Mi + Nj$ is a gradient field. To show $F$ is a gradient field, we must show that $F$ is continuously differentiable and $M_y = N_x$ for all $x, y$.

   Indeed, $F$ is continuously differentiable for all $x, y$ by inspection. Here $M = xe^x + y$ and $N = x$, so $M_y = N_x = 1$. We conclude that $(xe^x + y) \, dx + x \, dy$ is exact.

3. Compute the two dimensional curl of $F$ for each of the vector fields below.
   a) $F = (x, xe^x + y)$
   b) $F = i + j$
   c) $F = (xy^2, x^2y)$

   **Answer:** We know that if $F = Mi + Nj$ then $\text{curl}F = N_x - M_y$.

   a) $\text{curl}F = (e^x + xe^x) - 0 = e^x(1 + x)$.

   (This looks similar to the conservative vector fields from previous problems, but its components have been swapped.)

   b) $M = N = 1$ so $\text{curl}F = 0 - 0 = 0$.

   c) $N_x = 2xy$ and $M_y = 2yx$, so $\text{curl}F = 0$. 

