Green’s Theorem

We start with the ingredients for Green’s theorem.
(i) \( C \) a simple closed curve (simple means it never intersects itself)
(ii) \( R \) the interior of \( C \).
We also require that \( C \) must be positively oriented, that is, it must be traversed so its interior is on the left as you move in around the curve. Finally we require that \( C \) be piecewise smooth. This means it is a smooth curve with, possibly a finite number of corners.

Here are some examples.

\[
\begin{align*}
\mathbf{F} &= \langle M, N \rangle \\
\oint_C M \, dx + N \, dy &= \iint_R N_x - M_y \, dA.
\end{align*}
\]

We call \( N_x - M_y \) the two dimensional curl and denote it \( \text{curl} \, \mathbf{F} \).

We can write also Green’s theorem as
\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl} \, \mathbf{F} \, dA.
\]

Example 1: (use the right hand side (RHS) to find the left hand side (LHS))
Use Green’s Theorem to compute
\[
I = \oint_C 3x^2y^2 \, dx + 2x^2(1 + xy) \, dy \quad \text{where } C \text{ is the circle shown.}
\]

By Green’s Theorem \( I = \iint_R 6x^2y + 4x - 6x^2y \, dA = 4 \iint_R x \, dA \).

We could compute this directly, but we know \( x_{cm} = \frac{1}{A} \iint_R x \, dA = a \)
\[
\Rightarrow \iint_R x \, dA = \pi a^3 \Rightarrow I = 4\pi a^3.
\]

Example 2: (Use the LHS to find the RHS.)
Use Green’s Theorem to find the area under one arch of the cycloid
\[
x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta).
\]
The picture shows the curve $C = C_1 - C_2$ surrounding the area we want to find. (Note the minus sign on $C_2$.) By Green’s Theorem,
\[ \oint_C -y \, dx = \iint_R dA = \text{area}. \]
Thus,
\[
\text{area} = \oint_{C_1 - C_2} -y \, dx = \int_{C_1} 0 \cdot dx - \int_{C_2} -y \, dx = \int_0^{2\pi} a^2 (1 - \cos \theta)^2 \, d\theta = 3\pi a^2.
\]