Problems: Green’s Theorem

Calculate \( \int_C -x^2 y \, dx + xy^2 \, dy \), where \( C \) is the circle of radius 2 centered on the origin.

**Answer:** Green’s theorem tells us that if \( \mathbf{F} = (M, N) \) and \( C \) is a positively oriented simple closed curve, then
\[
\int_C M \, dx + N \, dy = \iint_R N_x - M_y \, dA.
\]

We let \( M = -x^2 y \) and \( N = xy^2 \) to get:
\[
\int_C -x^2 y \, dx + xy^2 \, dy = \iint_R y^2 - (-x^2) \, dA
\]
\[
= \iint_R x^2 + y^2 \, dA
\]
\[
= \int_0^{2\pi} \int_0^2 r^2 \, r \, dr \, d\theta
\]
\[
= \int_0^{2\pi} \frac{8}{3} \, d\theta
\]
\[
= \frac{16\pi}{3}.
\]

This result is \( 4/3 \) times the area \( \iint_R 1 \, dA \) of the circle, and so is a plausible answer.