Green’s Theorem: Sketch of Proof

**Green’s Theorem:** \[ \oint_C M \, dx + N \, dy = \iint_R N_x - M_y \, dA. \]

**Proof:**

i) First we’ll work on a rectangle. Later we’ll use a lot of rectangles to approximate an arbitrary region.

ii) We’ll only do \( \oint_C M \, dx \) (\( \oint_C N \, dy \) is similar).

By direct calculation the right hand side of Green’s Theorem

\[ \iint_R \frac{\partial M}{\partial y} \, dA = \int_a^b \int_c^d \frac{\partial M}{\partial y} \, dy \, dx. \]

Inner integral: \(-M(x,y) |_{c}^{d} = -M(x,d) + M(x,c)\)

Outer integral: \(\int_a^b \frac{\partial M}{\partial y} \, dA = \int_a^b M(x,c) - M(x,d) \, dx.\)

For the LHS we have

\[ \oint_C M \, dx = \int_{bottom} M \, dx + \int_{top} M \, dx \quad (\text{since } dx = 0 \text{ along the sides}) \]

\[ = \int_a^b M(x,c) \, dx + \int_b^a M(x,d) \, dx = \int_a^b M(x,c) - M(x,d) \, dx. \]

So, for a rectangle, we have proved Green’s Theorem by showing the two sides are the same.

In lecture, Professor Auroux divided \( R \) into “vertically simple regions”. This proof instead approximates \( R \) by a collection of rectangles which are especially simple both vertically and horizontally.

For line integrals, when adding two rectangles with a common edge the common edges are traversed in opposite directions so the sum is just the line integral over the outside boundary. Similarly when adding a lot of rectangles: everything cancels except the outside boundary. This extends Green’s Theorem on a rectangle to Green’s Theorem on a sum of rectangles. Since any region can be approximated as closely as we want by a sum of rectangles, Green’s Theorem must hold on arbitrary regions.