1. Base: $R: x^2 + (y - 1)^2 \leq 1$. Top: $z = f(x, y) = (x^2 + y^2)^{1/2}$.

In cylindrical coords, base is

$$
0 \leq r \leq 2 \sin \theta \\
0 \leq \theta \leq \pi
$$

The top is $z = r$.

(a)

$$
V = \int_{0}^{\pi} \int_{0}^{2 \sin \theta} \int_{0}^{r} 1 dz \ r dr \ d\theta = \int_{0}^{\pi} \int_{0}^{2 \sin \theta} r \cdot r dr \ d\theta
$$

$$
= \int_{0}^{\pi} r^3 / 3 \bigg|_{0}^{2 \sin \theta} d\theta = \frac{8}{3} \int_{0}^{\pi} \sin^3 \theta d\theta
$$

$$
= \frac{16}{3} \int_{0}^{\pi/2} \sin^3 \theta \ d\theta
$$

with the last step by symmetry. Then use table 113, $n = 3$ to get

$$
= \frac{16}{3} \cdot \frac{2}{3} = \frac{32}{9}.
$$
\[ \bar{z} = \frac{1}{V} \int \int \int_G z \, dV = \frac{1}{V} \int_0^{\pi} \int_0^{2\sin \theta} \int_0^r z \, dz \, r \, dr \, d\theta \]
\[ = \frac{1}{V} \int_0^{\pi} \int_0^{2\sin \theta} \left[ \frac{z^2}{2} \right]_{0}^{r} r \, dr \, d\theta \]
\[ = \frac{1}{V} \int_0^{\pi} \int_0^{2\sin \theta} \frac{r^3}{2} \, dr \, d\theta = \frac{2}{V} \int_0^{\pi} \sin^4 \theta \, d\theta \]
\[ = \frac{4}{V} \int_0^{\pi/2} \sin^4 \theta \, d\theta = \frac{4 \cdot 13 \pi}{V \cdot 32} = \frac{3 \pi}{4V}. \]

Now \( V = 32/9 \) so \( \bar{z} = \frac{27\pi}{248} \approx 0.6627 \) and this is approximately a third of the maximum height (=2) of \( G \).

2. Looking at the original integral, we see that \( G \) is the region
\[
0 \leq z \leq \sqrt{4 - x^2}, \quad 0 \leq y \leq 2x, \quad 0 \leq x \leq 2
\]

(a)

(b) We may re-describe \( G \) as
\[
0 \leq y \leq 2x, \quad 0 \leq x \leq \sqrt{4 - z^2}, \quad 0 \leq z \leq 2
\]
This gives
\[
\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_0^{2x} f(x, y, z) \, dy \, dx \, dz
\]

(c) We may redescribe \( G \) as
\[
y/2 \leq x \leq \sqrt{4 - z^2}, \quad 0 \leq z \leq \sqrt{4 - (y/2)^2}, \quad 0 \leq y \leq 4
\]
This gives
\[ \int_0^1 \int_0^{4-(y/z)^2} \int_{y/2}^{\sqrt{4-z^2}} f(x,y,z) \, dx \, dz \, dy. \]

(d) \( f = 1. \) Then the original integral is
\[ \int_0^2 \int_0^{2x} \int_0^{\sqrt{4-x^2}} \, dz \, dy \, dx = \int_0^2 \int_0^{2x} (4-x^2)^{1/2} \, dy \, dx = \int_0^2 2x(4-x^2)^{1/2} \, dx \]
\[ = -2/3 (4-x^2)^{3/2} \bigg|_0^2 = 16/3. \]

The integral from (b) is
\[ \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{2x} \, dy \, dx \, dz = \int_0^2 \int_0^{\sqrt{4-z^2}} 2x \, dx \, dz \]
\[ = \int_0^2 \left[ x^2 \right]_0^{(4-z^2)^{1/2}} \, dz = \int_0^2 (4-z^2) \, dz = [4z - z^3/3]_0^2 = 16/3. \]

The integral from (c) is
\[ \int_0^4 \int_0^{\sqrt{4-(y/z)^2}} \int_{y/2}^{\sqrt{4-z^2}} \, dx \, dz \, dy = \int_0^4 \int_0^{\sqrt{4-(y/2)^2}} (\sqrt{4-z^2} - y/2) \, dz \, dy = \int_0^4 \int_0^{\sqrt{4-(y/2)^2}} \sqrt{4-z^2} \, dz \, dy - \int_0^4 \int_0^{\sqrt{4-(y/2)^2}} y/2 \, dz \, dy \]
Trig sub’s and/or tables would be needed in order to get these to work out in exact form.

3. If \( V \) is the volume of the full vessel, we have that
\[ \text{Energy} = \int \int \int_V \, dE \]
where
\[ dE = g \, z \, dm = 10^3 \, g \, z \, dV \]
since the density of water is \( 10^3 \). Therefore we compute:
\[ \text{Energy} = \int \int \int_V 10^3 \, g \, z \, dV = \int_0^{100} \int_{-9}^{81} \int_{1/8\pi}^{8\pi} 10^3 \, g \, z \, dz \, dx \, dy = \ldots = 5.14 \times 10^{10} \text{joules}. \]
18.02SC Multivariable Calculus
Fall 2010

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.