Triple Integrals

1. Find the moment of inertia of the tetrahedron shown about the $z$-axis. Assume the tetrahedron has density $1$.

![Tetrahedron Diagram]

Figure 1: The tetrahedron bounded by $x + y + z = 1$ and the coordinate planes.

**Answer:** To compute the moment of inertia, we integrate distance squared from the $z$-axis times mass:

$$
\iiint_D (x^2 + y^2) \cdot 1 \, dV.
$$

Using cylindrical coordinates about the axis of rotation would give us an “easy” integrand ($r$) with complicated limits. The integrand $x^2 + y^2$ is not particularly intimidating, so we instead use rectangular coordinates. Integrating first with respect to $y$ or $x$ is preferable; $(x^2 + y^2)(1 - x - y)$ is a somewhat more intimidating integrand.

To find our limits of integration, we let $y$ go from $0$ to the slanted plane $x + y + z = 1$. The $x$ and $z$ coordinates are in $R$, the projection of $D$ to the $xz$-plane which is bounded by the $x$ and $z$ axes and the line $x + z = 1$.

\[
\text{Moment of Inertia} = \int_0^1 \int_0^{1-x-z} \int_0^{1-x-z} (x^2 + y^2) \, dy \, dx \, dz.
\]

Inner: \((x^2y + \frac{1}{3}y^3)^{1-x-z}\) \(= x^2 - x^3 - x^2z + \frac{1}{3}(1-x-z)^3\).

Middle:\n\[
\int_0^{1-z} x^2(1-z) - x^3 + \frac{1}{3}(1-x-z)^3 \, dx = \left[ \frac{1}{3}x^3(1-z) - \frac{1}{4}x^4 - \frac{1}{12}(1-x-z)^4 \right]_0^{1-z} = \frac{1}{3}(1-z)^4 - \frac{1}{4}(1-z)^4 + \frac{1}{12}(1-z)^4 = \frac{1}{6}(1-z)^4.
\]

Outer: \(\frac{1}{30}(1-z)^5 \bigg|_0^1 = \frac{1}{30}\).

2. Find the mass of a cylinder centered on the $z$-axis which has height $h$, radius $a$ and density $\delta = x^2 + y^2$. 


Answer: To find the mass we integrate the product of density and volume:

$$\text{Mass} = \iiint_D \delta \, dV = \iiint_D r^2 \, dV.$$  

Naturally, we’ll use cylindrical coordinates in this problem. The limits on $z$ run from 0 to $h$. The $x$ and $y$ coordinates lie in a disk of radius $a$, so $0 \leq r \leq a$ and $0 < \theta \leq 2\pi$.

$$\text{Mass} = \iiint_D r^2 \, dV = \int_0^{2\pi} \int_0^a \int_0^h r^2 \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^a \int_0^h r^3 \, dz \, dr \, d\theta.$$

Inner integral: $r^3 z|_0^h = hr^3$.

Middle integral: $\int_0^a hr^3 \, dr = \frac{ha^4}{4}$.

Outer integral: $2\pi \frac{ha^4}{4} = \frac{\pi ha^4}{2}$. 