Problems: Applications of Spherical Coordinates

Find the average distance of a point in a solid sphere of radius $a$ from:

(a) the center,
(b) a fixed diameter, and
(c) a fixed plane through the center.

**Answer:** Recall that the average value of a function $f(x, y, z)$ over a volume $D$ is given by $\frac{1}{V} \iiint_{D} f(x, y, z) \, dV$. We know $V = \frac{4}{3} \pi a^3$. For each of these problems, we’ll assume $D$ is a sphere centered at the origin.

(a) In this case, $f(x, y, z) = \rho$ and so:

$$\text{A.V.} = \frac{1}{4\pi a^3/3} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{a} \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{4\pi a^3/3} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{a} \frac{1}{4} a^4 \sin \phi \, d\phi \, d\theta$$

$$= \frac{3a}{16\pi} \int_{0}^{2\pi} 2 \, d\theta$$

$$= \frac{3a}{8\pi} (2\pi) = 3a/4.$$

(b) Here we’ll use the $z$-axis as the diameter in question, in which case $f = r = \rho \sin \phi$.

$$\text{A.V.} = \frac{1}{4\pi a^3/3} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{a} \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{3}{4\pi a^3} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{1}{4} a^4 \sin^2 \phi \, d\phi \, d\theta$$

$$= \frac{3a}{16\pi} \int_{0}^{2\pi} \frac{\pi}{2} \, d\theta$$

$$= \frac{3a}{32} \cdot (2\pi) = 3\pi a/16.$$

(c) If we choose the $xy$-plane, $f = |z|$. Because spheres are symmetric the average value of the upper half will equal the average value over the whole sphere, so we compute just that ($V = \frac{2}{3} \pi a^3$).

$$\text{A.V.} = \frac{3}{2\pi a^3} \int_{0}^{2\pi} \int_{0}^{\pi/2} \int_{0}^{a} \rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{3}{2\pi a^3} \int_{0}^{2\pi} \int_{0}^{\pi/2} \frac{1}{4} a^4 \cos \phi \sin \phi \, d\phi \, d\theta$$

$$= \frac{3a}{8\pi} \int_{0}^{2\pi} \frac{1}{2} \, d\theta$$

$$= \frac{3a}{16\pi} (2\pi) = 3a/8.$$