Problems: Vector Fields in Space

Find the gravitational attraction of an upper solid half-sphere of radius $a$ and center $(0,0,0)$ on a mass $m_0$ at $(0,0,0)$. Assume this half-sphere has density $\delta = z$.

**Answer:** Draw a picture.

We follow the steps outlined in recitation, changing only the density.

The force is $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$. (This notation is unfortunately standard. The subscript indicates component, not partial derivative.) By symmetry we know $F_x = F_y = 0$.

At $(x,y,z)$ a small volume $dV$ has mass $dm = \delta(x,y,z) dV = z dV$. This mass $dm$ exerts a force $\frac{Gm_0 \, dm}{\rho^3} \langle x, y, z \rangle$ on the test mass. The $z$-component of this force is $\frac{z G m_0 \, dm}{\rho^3}$, so $F_z = \iiint_D \frac{z G m_0 \, z \, dV}{\rho^3}$.

The limits in spherical coordinates are: $\rho$ from 0 to $a$, $\phi$ from 0 to $\pi/2$, $\theta$ from 0 to $2\pi$.

Recall that $z = \rho \cos \phi$. Then:

$$F_z = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a G m_0 \frac{z^2}{\rho^3} dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a G m_0 \frac{\rho^2 \cos^2 \phi}{\rho^3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a G m_0 \rho \cos^2 \phi \sin \phi \, d\rho \, d\phi \, d\theta.$$

**Inner integral:** $\frac{G m_0 a^2}{2} \cos^2 \phi \sin \phi$

**Middle integral:** $\frac{G m_0 a^2}{2} \left( -\frac{\cos^3 \phi}{3} \right) \bigg|_0^{\pi/2} = \frac{G m_0 a^2}{6}$

**Outer integral:** $\frac{G m_0 \pi a^2}{3} \Rightarrow \mathbf{F} = (0,0,Gm_0 \pi a^2 / 3)$.

Surprisingly, this is the same as the force exerted by a half-sphere with density $\sqrt{x^2 + y^2}$. 