Problems: Flux Through a Paraboloid

Consider the paraboloid \( z = x^2 + y^2 \). Let \( S \) be the portion of this surface that lies below the plane \( z = 1 \). Let \( \mathbf{F} = x\mathbf{i} + y\mathbf{j} + (1 - 2z)\mathbf{k} \).

Calculate the flux of \( \mathbf{F} \) across \( S \) using the outward normal (the normal pointing away from the \( z \)-axis).

**Answer:** First, draw a picture:

![Diagram of a paraboloid](image)

The surface \( S \) is a bowl centered on the \( z \)-axis. The outward normal \( \mathbf{n} \) points away from the outside of the bowl and downward. The region \( R \) is the shadow of the bowl – the unit circle in the \( xy \)-plane.

We know the \( z \) component of \( \mathbf{n} \) is negative, so \( \mathbf{n} \, dS = \langle z_x, z_y, -1 \rangle \, dx \, dy = \langle 2x, 2y, -1 \rangle \, dx \, dy \).

Thus, \( \mathbf{F} \cdot \mathbf{n} \, dS = (2x^2 + 2y^2 + 2z - 1) \, dx \, dy = (4z - 1) \, dx \, dy = (4r^2 - 1) \, dx \, dy \).

\[
\begin{align*}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R (4r^2 - 1) \, dx \, dy \\
&= \int_0^{2\pi} \int_0^1 (4r^2 - 1) r \, dr \, d\theta \\
&= \int_0^{2\pi} \left. \frac{1}{2} r^2 \right|_0^1 \, d\theta \\
&= \frac{\pi}{2} \\
&= \frac{\pi}{2}.
\end{align*}
\]
18.02SC Multivariable Calculus
Fall 2010

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